Systems of Ternariants that are Algebraically Complete.

By A. R. Forsyth, M.A., F.R.S., *Fellow of Trinity College, Cambridge.*

(Continued from page 60.)

PART III.

Applications to Bipartite Quantics.

59. The theory given in Part I holds alike for unipartite and for bipartite quantics: the difference in details arises through the difference of the literal operators. If in the quantic, symbolically represented by $a_x^m u_a^\mu$, the coefficient of $x_1^r x_2^s x_3^t \, u_1^\rho u_2^\sigma u_3^\tau$ be

$$\frac{m!}{r!\,s!\,t!}\,\frac{\mu!}{\rho!\,\sigma!\,\tau!}\,a_{r,s,t,\rho,\sigma,\tau},$$

(with the conditions $r+s+t=m$, $\rho+\sigma+\tau=\mu$), then the six operators similar to those of §1 are*

$$\left.\begin{aligned}
D_3 &= \Sigma\left(ra_{r-1,s+1,t,\rho,\sigma,\tau} - \sigma a_{r,s,t,\rho+1,\sigma-1,\tau}\right)\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}\\
D_1 &= \Sigma\left(sa_{r,s-1,t+1,\rho,\sigma,\tau} - \tau a_{r,s,t,\rho,\sigma+1,\tau-1}\right)\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}\\
D_2 &= \Sigma\left(ta_{r+1,s,t-1,\rho,\sigma,\tau} - \rho a_{r,s,t,\rho-1,\sigma,\tau+1}\right)\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}\\
D_4 &= \Sigma\left(sa_{r+1,s-1,t,\rho,\sigma,\tau} - \rho a_{r,s,t,\rho-1,\sigma+1,\tau}\right)\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}\\
D_6 &= \Sigma\,(ta_{r,s+1,t-1,\rho,\sigma,\tau} - \sigma a_{r,s,t,\rho,\sigma-1,\tau+1}\;\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}\\
D_5 &= \Sigma\left(ra_{r-1,s,t+1,\rho,\sigma,\tau} - \tau a_{r,s,t,\rho+1,\sigma,\tau-1}\right)\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}
\end{aligned}\right\} \quad (1').$$

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* See the memoir there cited, p. 42, where the signs + within the bracket should be changed to —.

I.—*The Lineo-Linear Quantic.*

60. In the symbolical form this is $a_x u_a$;* the explicit form we shall take to be

$$U = (a_1 x_1 + h_1 x_2 + g_1 x_3) u_1 + (a_2 x_1 + h_2 x_2 + g_2 x_3) u_2 + (a_3 x_1 + h_3 x_2 + g_3 x_3) u_3;$$

the sequence of literal coefficients and the arrangement of numerical subscripts

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* The principal memoirs dealing with the theory of bilinear forms are given in the following list:

Weierstrass (1858). "Ueber ein die homogenen Functionen zweiten Grades betreffendes Theorem." Berl. Monatsb., 1858, pp. 207–220.

Kronecker (1866). "Ueber bilineare Formen." Berl. Monatsb., 1866, pp. 597–612; reprinted in Crelle, t. LXVIII (1868), pp. 273–285.

Weierstrass (1868). "Zur Theorie der bilinearen und quadratischen Formen." Berl. Monatsb., 1868, pp. 310–338.

Kronecker (1868). Bemerkungen zu vorstehendem Vortrag. Ib., pp. 339–346.

Christoffel (1868). "Theorie der bilinearen Functionen." Crelle, t. LXVIII, pp. 253–272.

Clebsch und Gordan (1869). "Ueber biternäre Formen mit contragredienten Variabeln." Math. Ann., t. i, pp. 359–400; specially pp. 371–400.

Beltrami (1873). "Sulle funzioni bilineari." Batt. Giorn., t. XI, pp. 98–106.

Jordan (1873). "Sur les polynômes bilinéaires." Comptes Rendus, t. LXXVII, pp. 1487–1491; Liouville, 2e Sér., t. XIX, pp. 35–54.

Jordan (1874). "Sur la réduction des formes bilinéaires." Comptes Rendus, t. LXXVIII, pp. 614–617. "Sur les systèmes de formes quadratiques." Comptes Rendus, t. LXXVIII, pp. 1763–1767.

Kronecker (1874). "Ueber Schaaren von quadratischen Formen." Berl. Monatsb., 1874, pp. 59–76. Nachtrag zu diesem Aufsatze. Ib., pp. 149–156. "Ueber Schaaren von quadratischen und bilinearen Formen." Ib., pp. 206–232. "Ueber die congruenten Transformationen der bilinearen Formen." Ib., pp. 397–447.

Darboux (1874). "Mémoire sur la théorie algébrique des formes quadratiques." Liouville, 2e Sér., t. XIX, pp. 347–396.

Jordan (1874). "Mémoire sur la réduction et la transformation des systèmes quadratiques." Ib., pp. 397–422.

Frobenius (1878). "Ueber lineare Substitutionen und bilineare Formen." Crelle, t. LXXXIV, pp. 1–63.

Bachmann (1873). "Untersuchungen über quadratischen Formen." Crelle, t. LXXVI, pp. 331–341.

Hermite (1874). Extrait d'un lettre à M. Borchardt. Crelle, t. LXXVIII, pp. 325–328.

Cayley. "Sur la transformation d'une fonction quadratique en elle-même par des substitutions linéaires." Crelle, t. L, pp. 288–299.

Stickelberger (1879). "Ueber Schaaren von bilinearen und quadratischen Formen." Crelle, t. LXXXVI, pp. 20–43.

Frobenius (1879). "Ueber die schiefe Invarianten einer bilinearen oder quadratischen Form." Ib., pp. 44–71.

Cayley (1858). "A memoir on the automorphic linear transformation of a bipartite quadric function." Phil. Trans., 1858, pp. 39–46.

Other references will be found in these memoirs which deal very largely with the transformation and canonization of the forms; the memoirs of Christoffel, Clebsch and Gordan, and the last by Frobenius, deal with the concomitants.

will be seen to harmonize with those in later examples. The six operators in (1′) are

$$D_3 = h_1 \frac{\partial}{\partial a_1} + h_2 \frac{\partial}{\partial a_2} + h_3 \frac{\partial}{\partial a_3} - a_1 \frac{\partial}{\partial a_2} - h_1 \frac{\partial}{\partial h_2} - g_1 \frac{\partial}{\partial g_2},$$
$$D_1 = g_1 \frac{\partial}{\partial h_1} + g_2 \frac{\partial}{\partial h_2} + g_3 \frac{\partial}{\partial h_3} - a_2 \frac{\partial}{\partial a_3} - h_2 \frac{\partial}{\partial h_3} - g_2 \frac{\partial}{\partial g_3},$$
$$D_2 = a_1 \frac{\partial}{\partial g_1} + a_2 \frac{\partial}{\partial g_2} + a_3 \frac{\partial}{\partial g_3} - a_3 \frac{\partial}{\partial a_1} - h_3 \frac{\partial}{\partial h_1} - g_3 \frac{\partial}{\partial g_1},$$
$$D_4 = a_1 \frac{\partial}{\partial h_1} + a_2 \frac{\partial}{\partial h_2} + a_3 \frac{\partial}{\partial h_3} - a_2 \frac{\partial}{\partial a_1} - h_2 \frac{\partial}{\partial h_1} - g_2 \frac{\partial}{\partial g_1},$$
$$D_6 = h_1 \frac{\partial}{\partial g_1} + h_2 \frac{\partial}{\partial g_2} + h_3 \frac{\partial}{\partial g_3} - a_3 \frac{\partial}{\partial a_2} - h_3 \frac{\partial}{\partial h_2} - g_3 \frac{\partial}{\partial g_2},$$
$$D_5 = g_1 \frac{\partial}{\partial a_1} + g_2 \frac{\partial}{\partial a_2} + g_3 \frac{\partial}{\partial a_3} - a_1 \frac{\partial}{\partial a_3} - h_1 \frac{\partial}{\partial h_3} - g_1 \frac{\partial}{\partial g_3}.$$

From the general theory it follows that every concomitant has its leading coefficient a common solution of $D_1 = 0$ and $D_6 = 0$. Proceeding therefore to obtain these solutions, we first consider the set of equations subsidiary to $D_1 = 0$ given by

$$\frac{da_1}{0} = \frac{dg_1}{0} = \frac{da_2}{0} = \frac{dg_2}{0} = \frac{dh_1}{g_1} = \frac{dh_2}{g_2} = \frac{dh_3}{g_3 - h_2} = \frac{da_3}{-a_2} = \frac{dg_3}{-g_2},$$

eight in all; eight independent integrals of them will therefore be required. We have

$$\theta_1 = a_1,$$
$$\theta_2 = a_2,$$
$$\theta_3 = g_1,$$
$$\theta_4 = g_2;$$

and we apply the process of §35. Changing D_6 to Δ, we have

$$\Delta\theta_1 = 0,$$

so that θ_1 is a solution common to the two equations. Now for the 'variable of reference' we might take either a_2 or g_1; taking the latter, we have θ_3 as the connecting variable of the modified equations in Δ. We have

$$\Delta\theta_3 = h_1, \quad \Delta\theta_2 = -a_3,$$

so that
$$\theta_3\Delta\theta_2 - \theta_2\Delta\theta_3 = -(a_3 g_1 + a_2 h_1) = -\theta_5,$$

this quantity θ_5 being easily verified to be another solution of the subsidiary set. And then

$$\Delta\theta_5 = 0,$$

so that θ_5 is a solution common to $D_1 = 0$ and $D_6 = 0$.

Next we have $$\Delta\theta_4 = h_2 - g_3,$$
so that $$\theta_3\Delta\theta_4 - 2\theta_4\Delta\theta_3 = g_1(h_2 - g_3) - 2h_1g_2 = 2\theta_6,$$
where θ_6 is easily verified to be another solution of the subsidiary set. Next we have $\Delta\theta_6 = -\frac{1}{2}h_1(h_2 - g_3) - h_3g_1$, and therefore

$$\theta_3\Delta\theta_6 - \theta_6\Delta\theta_3 = -h_3g_1^2 + g_1h_1(g_3 - h_2) + g_2h_1^2 = \theta_7,$$

where θ_7 is a solution of the subsidiary set. And then

$$\Delta\theta_7 = 0,$$

so that θ_7 is a solution common to $D_1 = 0$ and $D_6 = 0$.

We now have seven integrals of the subsidiary set; the remaining one necessary we may take to be

$$\theta_8 = h_2 + g_3;$$

and we have $$\Delta\theta_8 = 0,$$

so that θ_8 is a solution common to the two characteristic equations.

It is easy to see that the eight integrals thus chosen are independent of one another.

61. Four common solutions of the two characteristic equations are $\theta_1, \theta_5, \theta_7, \theta_8$; for the remaining two that (§35) are necessary, we have

$$\theta_3^2\Delta(\theta_2 \div \theta_3) = -\theta_5;$$
$$\theta_3^2\Delta(\theta_4 \div \theta_3^2) = 2\theta_6 \div \theta_3,$$
$$\theta_3^2\Delta(\theta_6 \div \theta_3) = \theta_7,$$

of which two solutions are

$$\phi_1 = \frac{\theta_6^2 - \theta_4\theta_7}{\theta_3^2} = \frac{1}{4}\theta_8^2 + g_2h_3 - g_3h_2,$$

and $$\phi_2 = \frac{\theta_5\theta_6 + \theta_2\theta_7}{\theta_3} = -h_3a_2g_1 + \frac{1}{2}(g_3 - h_2)(a_2h_1 - a_3g_1) - g_2a_3h_1.$$

There are thus six common solutions, the values of which in terms of the coefficients of the quantic are

$$\theta_1 = v_0 = a_1,$$
$$\theta_5 = v_1 = (a_3, a_2)(g_1, h_1),$$
$$\theta_7 = v_2 = (-h_3, g_3 - h_2, g_2)(g_1, h_1)^2,$$
$$\theta_8 = g_3 + h_2;$$

instead of ϕ_1, which is practically the discriminant of v_2 considered as a binary form, we take

$$\vartheta_2 = \phi_1 - \frac{1}{4}\theta_8^2 = g_2 h_3 - g_3 h_2;$$

and instead of ϕ_2, which is the Jacobian of v_1 and v_2 considered as binary forms in g_1 and h_1 as variables, we take

$$\begin{aligned}\vartheta_3 &= \phi_2 + \frac{1}{2}\theta_5\theta_8 \\ &= a_2(g_3 h_1 - g_1 h_3) + a_3(g_1 h_2 - g_2 h_1).\end{aligned}$$

To obtain the order and the class to be associated with each of these quantities as leading coefficients of a concomitant, we use (I) and (II) and easily find,

$$\begin{aligned}\Theta_1 &= \theta_1 x_1 u_1 + \ldots\ldots, \\ \Theta_5 &= \theta_5 x_1^2 u_1^2 + \ldots\ldots, \\ \Theta_7 &= \theta_7 u_1^3 \quad + \ldots\ldots, \\ \Theta_8 &= \theta_8 x_1 u_1 + \ldots\ldots, \\ \Theta_2 &= \vartheta_2 x_1 u_1 + \ldots\ldots, \\ \Theta_3 &= \vartheta_3 x_1^2 u_1^2 + \ldots\ldots\end{aligned}$$

Further, we at once have

$$\theta_1 + \theta_8 = I_1,$$

an invariant, and $$u_x I_1 = \Theta_1 + \Theta_8;$$

and also $$\theta_1\vartheta_2 + \vartheta_3 = \begin{vmatrix} a_1, & a_2, & a_3 \\ g_1, & g_2, & g_3 \\ h_1, & h_2, & h_3 \end{vmatrix} = I_3,$$

an invariant, and $$u_x^2 I_3 = \Theta_1\Theta_2 + \Theta_3.$$

Another invariant, of the second degree, is given in §63.

It follows from the general theory that *every concomitant of the lineo-linear ternary quantic can be expressed in terms of the six independent concomitants* Θ_1, Θ_5, Θ_7, Θ_8, Θ_2, Θ_3.

62. But if, instead of taking in §60 the quantity θ_3 as the variable of reference, we take θ_2 as that variable, we are led to the following system of independent solutions common to the two characteristic equations:

$$\begin{aligned}\theta_1 &= a_1, \\ \theta_5 &= (h_1,\ g_1 \between a_2,\ a_3), \\ \theta_8 &= g_3 + h_2, \\ \vartheta_2 &= g_2 h_3 - g_3 h_2, \\ \vartheta_3 &= a_2(g_3 h_1 - g_1 h_3) + a_3(g_1 h_2 - g_2 h_1), \\ \phi_7 &= (-h_3,\ h_2 - g_3,\ g_2 \between a_2,\ a_3)^2,\end{aligned}$$

the first five of which are the same as before and necessarily lead to the same concomitants, while the sixth leads to the concomitant

$$\Phi_7 = \phi_7 x_1^3 + \ldots .$$

Hence (§61, fin.) Φ_7 must be expressible in terms of $\Theta_1, \Theta_5, \Theta_7, \Theta_8, \Theta_2, \Theta_3$.

63. Now Clebsch and Gordan have already* given the complete system of asyzygetic concomitants of the bilinear ternary quantic; they are, in addition to u_x,

Symbol (*C.* and *G.*)	Symbolic Form.	Evaluate Form.
f	$= a_x u_\alpha$	$= a_1 x_1 u_1 + \ldots ,$
i	$= a_\alpha$	$= a_1 + h_2 + g_3,$
i_1	$= a_\beta b_\alpha$	$= a_1^2 + h_2^2 + g_3^2 + 2a_2 h_1 + 2a_3 g_1 + 2g_2 h_3,$
f_1	$= a_x u_\beta b_\alpha$	$= (a_1^2 + a_2 h_1 + a_3 g_1) x_1 u_1 + \ldots ,$
i_2	$= a_\beta b_\gamma c_\alpha$	$= \frac{3}{2} i i_1 - \frac{1}{2} i_1^3 + \begin{vmatrix} a_1, & a_2, & a_3 \\ h_1, & h_2, & h_3 \\ g_1, & g_2, & g_3 \end{vmatrix},$
ϕ	$= a_x c_x b_\alpha (\beta\gamma x)$	$= \{- a_2^2 h_3 + a_2 a_3 (h_2 - g_3) + g_2 a_3^2\} x_1^3 + \ldots ,$
ψ	$= u_\beta u_\gamma b_\alpha (acu)$	$= \{g_1^2 h_3 + g_1 h_1 (h_2 - g_3) - h_1^2 g_2\} u_1^3 + \ldots .$

From the theorem that the system $\Theta_1, \Theta_5, \Theta_7, \Theta_8, \Theta_2, \Theta_3$ is algebraically complete, it follows that each one of this set can be expressed in terms of members of that system; and, in fact, it is easy to prove the relations:

$$\begin{aligned}
f &= \Theta_1, \\
u_x i &= \Theta_1 + \Theta_8, \\
u_x^2 i_1 &= \Theta_1^2 + \Theta_8^2 + 2\Theta_5 + 2u_x\Theta_2, \\
u_x f_1 &= \Theta_1^2 + \Theta_5, \\
u_x^2 i_2 &= u_x^2 \left(\frac{3}{2} i i_1 - \frac{1}{2} i^3\right) - \Theta_1\Theta_2 - \Theta_3, \\
\phi &= \Phi_7, \\
\psi &= - \Theta_7 .
\end{aligned}$$

And the expression of Φ_7 in terms of the system of six concomitants Θ is easily shown to be given by

$$u_x^2 \Theta_7 \Phi_7 = u_x \Theta_3^2 - \Theta_3 \Theta_5 \Theta_8 - \Theta_2 \Theta_5^2 .$$

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*See pp. 372 et seq. of their memoir, quoted in the note to §60.

This equation is, after the foregoing relations, equivalent to an equation among Clebsch and Gordan's forms; the verification of this result is easily obtainable from the canonical forms of the concomitants (p. 386 l. c.).

64. If, in the two equations of §60 which determine leading coefficients, we take

$$g_3 - h_2 = 2k, \quad g_3 + h_2 = \theta_8,$$

we find

$$D_1 = g_1 \frac{\partial}{\partial h_1} - a_2 \frac{\partial}{\partial a_3} - g_2 \frac{\partial}{\partial k} + 2k \frac{\partial}{\partial h_3} = 0,$$

$$D_6 = h_1 \frac{\partial}{\partial g_1} - a_3 \frac{\partial}{\partial a_2} + h_3 \frac{\partial}{\partial k} - 2k \frac{\partial}{\partial g_2} = 0.$$

When these are written in the forms

$$g_1 \frac{\partial}{\partial h_1} - a_2 \frac{\partial}{\partial a_3} = \quad g_2 \frac{\partial}{\partial k} - 2k \frac{\partial}{\partial h_3},$$

$$h_1 \frac{\partial}{\partial g_1} - a_3 \frac{\partial}{\partial a_2} = - h_3 \frac{\partial}{\partial k} + 2k \frac{\partial}{\partial g_2},$$

they are the differential equations of *the concomitants of a binary quadratic in literal coefficients* $-h_3$, k, g_2, *the variables in the concomitants being* g_1 and h_1, a_2 and $-a_3$.

To this system of solutions must be added a_1 and θ_8, neither of which enters into the transformed equations and both of which are therefore solutions.

This inference is verified immediately on a reference to the system of solutions obtained, the solution θ_5 being the determinant of the variables.

Similar inferences in the succeeding cases may be derived after the characteristic equations have been transformed by similar substitutions; but the inferences will be left unstated, because obvious from the respective systems of solutions, and the equations will not be transformed, the requisite substitutions being indicated sufficiently by the respective systems of solutions.

II.—*A System of Two Lineo-Linear Quantics.*

65. They may be taken in the forms

$$U = (a_1x_1 + h_1x_2 + g_1x_3)\,u_1 + (a_2x_1 + h_2x_2 + g_2x_3)\,u_2 + (a_3x_1 + h_3x_2 + g_3x_3)\,u_3,$$

$$U' = (a_1'x_1 + h_1'x_2 + g_1'x_3)\,u_1 + (a_2'x_1 + h_2'x_2 + g_2'x_3)\,u_2 + (a_3'x_1 + h_3'x_2 + g_3'x_3)\,u_3,$$

and the literal operators are now of the form $D_\lambda + D_\lambda'$ instead of D_λ. We must therefore find the simultaneous independent solutions of

$$D_1 + D_1' = 0, \quad \Delta = D_6 + D_6' = 0.$$

From the 17 equations, which are subsidiary to the solution of the former equation, it at once appears that there are eight integrals of the form

$$\left.\begin{array}{llll}\theta_1 = a_1, & \theta_2 = a_2, & \theta_3 = g_1, & \theta_4 = g_2 \\ \theta_1' = a_1', & \theta_2' = a_2', & \theta_3' = g_1', & \theta_4' = g_2'\end{array}\right\};$$

and these integrals are used for the deduction of further integrals by the usual method hitherto adopted, and also for the modification of the Δ-equation. Also, as in the case of a single lineo-linear quantic it was possible to take either θ_3 or θ_2 as variable of reference, it is now possible to take any one of the four θ_2, θ_3, θ_2', θ_3' as variable of reference.

As in the earlier case of the system of three quadratics (53–58), I merely give some important stages of the work and the results.

Since there are eighteen coefficients, it follows (§18) that *all the simultaneous solutions can be expressed in terms of fifteen independent simultaneous solutions.*

66. In addition to the preceding eight quantities θ, which are solutions of $D_1 + D_1' = 0$ and among which θ_1 and θ_1' are also solutions of $\Delta = 0$, the following quantities—all being solutions of $D_1 + D_1' = 0$—arise in the formation of the modified Δ-equations:

$$\left.\begin{array}{l}\theta_6 = (h_2 - g_3)\, g_1 - 2g_2h_1 \\ \theta_6' = (h_2' - g_3')\, g_1' - 2g_2'h_1'\end{array}\right\}, \quad \left.\begin{array}{l}\psi_6 = (h_2' - g_3')\, g_1 - 2g_2'h_1 \\ \psi_6' = (h_2 - g_3)\, g_1' - 2g_2h_1'\end{array}\right\};$$

$$\left.\begin{array}{l}\chi_6 = (h_2 - g_3)\, a_2 + 2g_2a_3 \\ \chi_6' = (h_2' - g_3')\, a_2' + 2g_2'a_3'\end{array}\right\}, \quad \left.\begin{array}{l}\phi_6 = (h_2' - g_3')\, a_2 + 2g_2'a_3 \\ \phi_6' = (h_2 - g_3)\, a_2' + 2g_2a_3'\end{array}\right\}.$$

And in the modified Δ-equations in the fourteen quantities, so far obtained yet not simultaneous solutions—viz. θ_2, θ_3, θ_4, θ_2', θ_3', θ_4' and the eight just given—the following further quantities arise, all of them being simultaneous solutions of the two characteristic equations:

$$\left.\begin{array}{l}\theta_5 = a_3g_1 + a_2h_1 \\ \theta_5' = a_3'g_1' + a_2'h_1'\end{array}\right\}, \quad \left.\begin{array}{l}\psi_5 = a_3'g_1 + a_2'h_1 \\ \psi_5' = a_3g_1' + a_2h_1'\end{array}\right\};$$

$$\phi = g_1h_1' - g_1'h_1; \qquad \psi = a_2'a_3 - a_3'a_2;$$

$$\left.\begin{array}{l}\theta_7 = (A,\ B,\ C \between g_1,\ -h_1)^2 \\ \lambda_7 = (A,\ B,\ C \between g_1,\ -h_1 \between g_1',\ -h_1') \\ \psi_7' = (A,\ B,\ C \between g_1',\ -h_1')^2\end{array}\right\}; \quad \left.\begin{array}{l}\psi_7 = (A',\ B',\ C' \between g_1,\ -h_1)^2 \\ \lambda_7' = (A',\ B',\ C' \between g_1,\ -h_1 \between g_1',\ -h_1') \\ \theta_7' = (A',\ B',\ C' \between g_1',\ -h_1')^2\end{array}\right\};$$

$$\left.\begin{array}{l}\chi_7 = (A,\ B,\ C \between a_2,\ a_3)^2 \\ \rho_7 = (A,\ B,\ C \between a_2,\ a_3 \between a_2',\ a_3') \\ \phi_7' = (A,\ B,\ C \between a_2',\ a_3')^2\end{array}\right\}; \quad \left.\begin{array}{l}\phi_7 = (A',\ B',\ C' \between a_2,\ a_3)^2 \\ \rho_7' = (A',\ B',\ C' \between a_2,\ a_3 \between a_2',\ a_3') \\ \chi_7' = (A',\ B',\ C' \between a_2',\ a_3')^2\end{array}\right\};$$

$$\left.\begin{aligned} \phi_2 &= (A, B, C \between g_1, -h_1 \between a_2, a_3) \\ \upsilon_2 &= (A, B, C \between g_1, -h_1 \between a_2', a_3') \\ \omega_2 &= (A, B, C \between g_1', -h_1' \between a_2, a_3) \\ \chi_2' &= (A, B, C \between g_1', -h_1' \between a_2', a_3') \end{aligned}\right\}; \qquad \left.\begin{aligned} \chi_2 &= (A', B', C' \between g_1, -h_1 \between a_2, a_3) \\ \omega_2' &= (A', B', C' \between g_1, -h_1 \between a_2', a_3') \\ \upsilon_2' &= (A', B', C' \between g_1', -h_1' \between a_2, a_3) \\ \phi_2' &= (A', B', C' \between g_1', -h_1' \between a_2', a_3') \end{aligned}\right\};$$

the quantities A, B, C and A', B', C' being

$$A = -2h_3', \quad B = h_2 - g_3, \quad C = 2g_2,$$
$$A' = -2h_3', \quad B' = h_2' - g_3', \quad C' = 2g_2'.$$

The following are the sets of modified Δ-equations in each of the four possible variables of reference; the first seven equations of each set are the independent equations in that set, but their aggregate in any one set is rendered (as in §47) complete in form by the introduction of the subsidiary quantities which occur in all the other sets. The equations are:

$\theta_3^2\Delta = \nabla_3$	$\theta_2^2\Delta = \nabla_2$	$\theta_3'^2\Delta = \nabla_3'$	$\theta_2^2\Delta = \nabla_2'$
$\nabla_3\epsilon = -\theta_5$	$\nabla_2\epsilon' = \theta_5$	$\nabla_3'\mu = -\theta_5'$	$\nabla_2'\mu' = \theta_5'$
$\nabla_3 p = q$	$\nabla_2 r = s$	$\nabla_3' p' = q'$	$\nabla_2' r' = s'$
$\nabla_3 q = \theta_7$	$\nabla_2 s = \chi_7$	$\nabla_3' q' = \theta_7'$	$\nabla_2' s' = \chi_7'$
$\nabla_3\gamma = \phi$	$\nabla_2\iota = \psi$	$\nabla_3'\delta' = -\psi_5'$	$\nabla_2'\iota' = -\psi$
$\nabla_3\alpha = -\psi_5$	$\nabla_2\delta = \psi_5'$	$\nabla_3'\gamma' = -\phi$	$\nabla_2'\alpha' = \psi_5$
$\nabla_3\pi = \varkappa$	$\nabla_2\rho = \phi_6$	$\nabla_3'\pi' = \varkappa'$	$\nabla_2'\rho' = \sigma'$
$\nabla_3\varkappa = \psi_7$	$\nabla_2\sigma = \phi_7$	$\nabla_3'\varkappa' = \psi_7'$	$\nabla_2'\sigma' = \phi_7'$
$\nabla_3 t = \phi_2$	$\nabla_2\xi = \phi_2$	$\nabla_3' u' = \lambda_7$	$\nabla_2'\zeta' = \upsilon_2$
$\nabla_3 l = \chi_2$	$\nabla_2\eta = \chi_2$	$\nabla_3' x' = \lambda_7'$	$\nabla_2'\omega' = \omega_2'$
$\nabla_3 u = \lambda_7'$	$\nabla_2\zeta = \upsilon_2'$	$\nabla_3'\tau' = \omega_2$	$\nabla_2' g' = \rho_7$
$\nabla_3 x = \lambda_7$	$\nabla_2\omega = \omega_2$	$\nabla_3'\lambda' = \upsilon_2'$	$\nabla_2' f' = \rho_7'$
$\nabla_3\tau = \omega_2'$	$\nabla_2 g = \rho_7'$	$\nabla_3' t' = \phi_2'$	$\nabla_2'\xi' = \phi_2'$
$\nabla_3\lambda = \upsilon_2$	$\nabla_2 f = \rho_7$	$\nabla_3' l' = \chi_2'$	$\nabla_2'\eta' = \chi_2'$

where the various quantities are defined by the relations

$$\left.\begin{array}{l}\theta_2 = \theta_3 \varepsilon \qquad\quad = \theta'_3 \delta' = \theta'_2 \iota' \\ \theta_3 \qquad\quad = \theta_2 \varepsilon' = \theta'_3 \gamma' = \theta'_2 \alpha'\end{array}\right\}, \quad \left.\begin{array}{l}\theta'_2 = \theta_3 \alpha = \theta_2 \iota = \theta'_3 \mu \\ \theta'_3 = \theta_3 \gamma = \theta_2 \delta \qquad\quad = \theta'_2 \mu'\end{array}\right\},$$

$$\left.\begin{array}{l}\theta_4 = \theta_3^2 p = \theta_2^2 r = \theta_3'^2 \pi' = \theta_2'^2 \rho' \\ \theta'_4 = \theta_3^2 \pi = \theta_2^2 \rho = \theta_3'^2 p' = \theta_2'^2 r'\end{array}\right\},$$

$$\left.\begin{array}{l}\theta_6 = \theta_3 q = \theta_2 \xi = \theta'_3 u' = \theta'_2 \zeta' \\ \theta'_6 = \theta_3 u = \theta_2 \zeta = \theta'_3 q' = \theta'_2 \xi'\end{array}\right\}, \quad \left.\begin{array}{l}\phi_6 = \theta_3 l = \theta_2 \sigma = \theta'_3 \lambda' = \theta'_2 f' \\ \phi'_6 = \theta_3 \lambda = \theta_2 f = \theta'_3 l' = \theta'_2 \sigma'\end{array}\right\},$$

$$\left.\begin{array}{l}\psi_6 = \theta_3 \varkappa = \theta_2 \eta = \theta'_3 v' = \theta'_2 \omega' \\ \psi'_6 = \theta_3 x = \theta_2 \omega = \theta'_3 \varkappa' = \theta'_2 \eta'\end{array}\right\}, \quad \left.\begin{array}{l}\chi_6 = \theta_3 t = \theta_2 s = \theta'_3 \tau' = \theta'_2 g' \\ \chi'_6 = \theta_3 \tau = \theta_2 g = \theta'_3 t' = \theta'_2 s'\end{array}\right\}.$$

67. The independent simultaneous solutions of the two characteristic equations can be derived from any one of the sets; let us, then, consider the first set, retaining for this purpose only the (first seven) independent equations. Regarding these equations as furnishing one system of common integrals, we have the eight solutions of $D_1 + D'_1 = 0$ before given, θ_1, θ_2, θ_3, θ_4 and θ'_1, θ'_2, θ'_3, θ'_4; and those constructed later, viz. θ_5; θ_6, θ_7; ϕ; ψ_5; ψ_6, ψ_7; being fifteen in all. Two more are necessary to make up the requisite number of seventeen solutions of $D_1 + D'_1 = 0$; and they (as in §60) may be taken

$$\theta_8 = h_2 + g_3,$$
$$\theta'_8 = h'_2 + g'_3.$$

Now θ_8 and θ'_8 both satisfy $\Delta = 0$, and therefore it appears that of the necessary fifteen simultaneous solutions we already have

$$\theta_1,\ \theta'_1;\ \theta_8,\ \theta'_8;\ \theta_5,\ \psi_5;\ \theta_7,\ \psi_7;\ \text{and } \phi,$$

so that six independent integrals—the proper number—must be obtained from the modified (∇_3-)equations. Six solutions, algebraically independent of one another and of those already obtained, are:

$$\begin{aligned}\varepsilon\theta_7 + q\theta_5 &= \phi_2, \\ q^2 - 2p\theta_7 &= (h_2 - g_3)^2 + 4h_3 g_2 = \Delta, \\ \varepsilon\phi + \gamma\theta_5 &= \psi'_5, \\ \alpha\theta_5 - \varepsilon\psi_5 &= \psi, \\ \varkappa^2 - 2\pi\psi_7 &= (h'_2 - g'_3)^2 + 4h'_3 g'_2 = \Delta', \\ \varepsilon\psi_7 + \varkappa\theta_5 &= \chi_2.\end{aligned}$$

Hence every simultaneous solution can be expressed in terms of the 15 (= 9 + 6) solutions already obtained.

68. These fifteen solutions are not, however, symmetrical with regard to the two quantics, and they will therefore be replaced by a system which is symmetrical and at the same time is algebraically equivalent to them. Among special relations—a fuller system will be given immediately—we have

$$\theta_5\theta_5' = \psi_5\psi_5' + \psi\phi,$$

so that we may replace ψ by θ_5'; we have

$$\begin{aligned}\theta_7'\psi_7 &= \lambda_7'^2 - \phi^2\Delta',\\ \lambda_7'\theta_5 &= \psi_7\psi_5' - \phi\chi_2,\end{aligned}$$

so that we may replace χ_2 by θ_7'; and the second of these equations we shall use with

$$\lambda_7'\theta_5' = \phi\phi_2' + \theta_7'\psi_5$$

to replace ψ_5 by ϕ_2'; and lastly, we have

$$\begin{aligned}\theta_7\psi_7' &= \lambda_7^2 - \phi^2\Delta,\\ \lambda_7\theta_5 &= \theta_7\psi_5' - \phi\phi_2,\end{aligned}$$

so that we may replace ψ_5' by ψ_7'.

The set is now constituted by

$$\left.\begin{matrix}\theta_1\\ \theta_1'\end{matrix}\right\}, \left.\begin{matrix}\theta_8\\ \theta_8'\end{matrix}\right\}, \left.\begin{matrix}\Delta\\ \Delta'\end{matrix}\right\}, \left.\begin{matrix}\theta_5\\ \theta_5'\end{matrix}\right\}, \left.\begin{matrix}\theta_7\\ \theta_7'\end{matrix}\right\}, \left.\begin{matrix}\phi_2\\ \phi_2'\end{matrix}\right\}, \psi_7, \psi_7', \phi,$$

a system symmetrical with regard to the two quantics; and it follows that *every simultaneous solution can be expressed in terms of this set.*

Two of the set, viz. Δ and Δ', may be simplified in form as in §61, for

$$\begin{aligned}\tfrac{1}{4}(\Delta - \theta_8^2) &= g_2h_3 - g_3h_2 = \vartheta_2,\\ \tfrac{1}{4}(\Delta' - \theta_8'^2) &= g_2'h_3' - g_3'h_2' = \vartheta_2',\end{aligned}$$

which will be taken as leading coefficients when the system of concomitants is established; but for the purpose of expressing dependent solutions it is convenient to retain Δ and Δ'. And as an intermediate quantity we have

$$\Delta_{12} = \varkappa q - p\psi_7 - \pi\theta_7 = (h_2 - g_3)(h_2' - g_3') + 2h_3'g_2 + 2h_3g_2',$$

a simultaneous solution of the two characteristic equations, which also admits of some simplification in form by taking

$$\vartheta_{12} = \tfrac{1}{2}(\Delta_{12} - \theta_8\theta_8') = g_2h_3' + g_2'h_3 - g_3h_2' - g_3'h_2\,;$$

but for the same purpose as Δ and Δ' it will be convenient to retain Δ_{12}.

69. Passing now to the consideration of the aggregate of modified Δ-equations in θ_3 as the variable of reference, we notice that eleven out of thirteen have their right-hand sides solutions of $\Delta = 0$, and that therefore, when any pair is combined, they lead to a solution; for instance, from

$$\nabla_3\varepsilon = -\theta_5, \quad \nabla_3 t = \phi_2$$

we have as a solution of $\nabla_3 = 0$, i. e. of $\Delta = 0$, the quantity $\varepsilon\phi_2 + t\theta_5$. The number of such combinations is 55 $\left(= \tfrac{1}{2}\,.\,11\,.\,10\right)$; and the corresponding 55 solutions can be expressed in terms of the fundamental set. Some of them are merely combinations of solutions already obtained, though not yet expressed in terms of the fundamental set; some of them are solutions new in form. The relations are given in the following table, which is to be read

$$\begin{aligned} &\varepsilon\theta_7 + (-q)(-\theta_5) = \phi_2, && \varepsilon\phi + (-\gamma)(-\theta_5) = \psi_5', \\ &q\phi + (-q)\,\theta_7 \quad = -\lambda_7', && q\psi_5 + \alpha\theta_7 \qquad\quad = v_2, \end{aligned}$$

and so on:

	$(\theta_7, -q)$	$(\phi, -\gamma)$	(ψ_5, α)	$(\psi_7, -\varkappa)$	$(\phi_2, -t)$	$(\chi_2, -l)$	$(\lambda_7', -u)$	$\lambda_7, -x$	$(\omega_2', -\tau)$	$(\upsilon_2, -\lambda)$
$(\varepsilon, -\theta_5)$	ϕ_2	ψ_5'	$-\psi$	χ_2	χ_7	ϕ_7	υ_2'	ω_2	ρ_7'	ρ_7
	(q, θ_7)	$-\lambda_7'$	υ_2	$\mathfrak{g}$	$\Delta\theta_5$	$\mathfrak{i}_1+\Delta_{12}\theta_5$	$\mathfrak{k}_{12}-\Delta_{12}\phi$	$-\Delta\phi$	$\mathfrak{i}_2+\Delta_{12}\psi_5$	$\Delta\psi_5$
		(γ, ϕ)	θ_5'	λ_7'	ω_2	υ_2	θ_7'	ψ_7'	ϕ_2'	χ_2'
			$(\alpha, -\psi_5)$	χ_2	ρ_7	ρ_7'	ϕ_2'	χ_2'	χ_7'	ϕ_7'
				$(\varkappa, \psi_7)$	$-\mathfrak{i}_1+\Delta_{12}\theta_5$	$\Delta'\theta_5$	$-\Delta'\phi$	$-\mathfrak{k}_{12}-\Delta_{12}\phi$	$\Delta'\psi_5$	$-\mathfrak{i}_2'+\Delta_{12}\psi_5$
					(t, ϕ_2)	$\mathfrak{k}$	$\mathfrak{i}_2'-\Delta_{12}\psi_5'$	$-\Delta\psi_5'$	$\mathfrak{g}_{12}-\Delta_{12}\psi$	$-\Delta\psi$
						(l, χ_2)	$-\Delta'\psi_5'$	$-\mathfrak{i}_2'-\Delta_{12}\psi_5'$	$-\Delta'\psi$	$-\mathfrak{g}_{12}-\Delta_{12}\psi$
							(u, λ_7')	$-\mathfrak{g}'$	$\Delta'\theta_5'$	$-\mathfrak{i}_1'+\Delta_{12}\theta_5'$
								(x, λ_7)	$\mathfrak{i}_1'+\Delta_{12}\theta_5'$	$\Delta\theta_5'$
									(τ, ω_2')	$-\mathfrak{k}'$

The quantities which occur in this table and not already defined are given by the following definitions, in which

$$\begin{aligned}
\mathfrak{A} &= -h_3'(h_2-g_3)+h_3(h_2'-g_3'),\\
\mathfrak{B} &= \quad g_2'h_3-g_2h_3',\\
\mathfrak{C} &= -g_2'(h_2-g_3)+g_2(h_2'-g_3');
\end{aligned}$$

it will be seen that all the functions are of the nature of Jacobians:

$$\left.\begin{aligned}
\mathfrak{g} &= 2(\mathfrak{A},\mathfrak{B},\mathfrak{C})(g_1,-h_1)^2\\
\mathfrak{g}_{12} &= 2(\mathfrak{A},\mathfrak{B},\mathfrak{C})(g_1,-h_1)(g_1',-h_1')\\
\mathfrak{g}' &= 2(\mathfrak{A},\mathfrak{B},\mathfrak{C})(g_1',-h_1')^2
\end{aligned}\right\},\qquad
\left.\begin{aligned}
\mathfrak{h} &= 2(\mathfrak{A},\mathfrak{B},\mathfrak{C})(a_2,a_3)^2\\
\mathfrak{h}_{12} &= 2(\mathfrak{A},\mathfrak{B},\mathfrak{C})(a_2,a_3)(a_2',a_3')\\
\mathfrak{h}' &= 2(\mathfrak{A},\mathfrak{B},\mathfrak{C})(a_2',a_3')^2
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\mathfrak{i}_1 &= 2(\mathfrak{A},\mathfrak{B},\mathfrak{C})(g_1,-h_1)(a_2,a_3)\\
\mathfrak{i}_1' &= 2(\mathfrak{A},\mathfrak{B},\mathfrak{C})(g_1',-h_1')(a_2',a_3')
\end{aligned}\right\},\qquad
\left.\begin{aligned}
\mathfrak{i}_2 &= 2(\mathfrak{A},\mathfrak{B},\mathfrak{C})(g_1,-h_1)(a_2,a_3')\\
\mathfrak{i}_2' &= 2(\mathfrak{A},\mathfrak{B},\mathfrak{C})(g_1',-h_1')(a_2,a_3)
\end{aligned}\right\}.$$

70. The simplest integral relations among the solutions already obtained are:

$$\left.\begin{aligned}
\lambda_7\theta_5 &= \theta_7\psi_5'-\phi\phi_2\\
\lambda_7\psi_5 &= \theta_7\theta_5'-\phi\upsilon_2\\
\lambda_7\psi_5' &= \psi_7'\theta_5+\phi\omega_2\\
\lambda_7\theta_5' &= \psi_7'\psi_5+\phi\chi_2'
\end{aligned}\right\};\qquad
\left.\begin{aligned}
\lambda_7'\theta_5 &= \psi_7\psi_5'-\phi\chi_2\\
\lambda_7'\psi_5 &= \psi_7\theta_5'-\phi\omega_2'\\
\lambda_7'\psi_5' &= \theta_7'\theta_5+\phi\upsilon_2'\\
\lambda_7'\theta_5' &= \theta_7'\psi_5+\phi\phi_2'
\end{aligned}\right\};$$

$$\left.\begin{aligned}
\rho_7\theta_5 &= \chi_7\psi_5+\psi\phi_2\\
\rho_7\psi_5 &= \phi_7'\theta_5-\psi\upsilon_2\\
\rho_7\psi_5' &= \chi_7\theta_5'+\psi\omega_2\\
\rho_7\theta_5' &= \phi_7'\psi_5'-\psi\chi_2'
\end{aligned}\right\},\qquad
\left.\begin{aligned}
\rho_7'\theta_5 &= \phi_7\psi_5+\psi\chi_2\\
\rho_7'\psi_5 &= \chi_7'\theta_5-\psi\omega_2'\\
\rho_7'\psi_5' &= \phi_7\theta_5'+\psi\upsilon_2'\\
\rho_7'\theta_5' &= \chi_7'\psi_5'-\psi\phi_2'
\end{aligned}\right\};$$

$$\left.\begin{aligned}
\theta_7\chi_7 &= \phi_2^2-\theta_5^2\Delta\\
\theta_7\phi_7' &= \upsilon_2^2-\psi_5^2\Delta\\
\psi_7'\chi_7 &= \omega_2^2-\psi_5'^2\Delta\\
\psi_7'\phi_7' &= \chi_2'^2-\theta_5'^2\Delta
\end{aligned}\right\};\qquad
\left.\begin{aligned}
\psi_7\phi_7 &= \chi_2^2-\theta_5^2\Delta'\\
\psi_7\chi_7' &= \omega_2'^2-\psi_5^2\Delta'\\
\theta_7'\phi_7 &= \upsilon_2'^2-\psi_5'^2\Delta'\\
\theta_7'\chi_7' &= \phi_2'^2-\theta_5'^2\Delta'
\end{aligned}\right\};$$

$$\left.\begin{aligned}
\theta_7\psi_7' &= \lambda_7^2-\phi^2\Delta\\
\theta_7'\psi_7 &= \lambda_7'^2-\phi^2\Delta'
\end{aligned}\right\};\qquad
\left.\begin{aligned}
\chi_7\phi_7' &= \rho_7^2-\psi^2\Delta\\
\chi_7'\phi_7 &= \rho_7'^2-\psi^2\Delta'
\end{aligned}\right\};$$

$$\left.\begin{aligned}
\mathfrak{g}\psi_5'-\mathfrak{g}_{12}\theta_5 &= \phi\mathfrak{i}_1\\
\mathfrak{g}\theta_5'-\mathfrak{g}_{12}\psi_5 &= \phi\mathfrak{i}_2\\
\mathfrak{g}_{12}\psi_5'-\mathfrak{g}'\theta_5 &= \phi\mathfrak{i}_2'\\
\mathfrak{g}_{12}\theta_5'-\mathfrak{g}'\psi_5 &= \phi\mathfrak{i}_1'
\end{aligned}\right\};\qquad
\left.\begin{aligned}
\mathfrak{h}'\theta_5-\mathfrak{h}_{12}\psi_5 &= \psi\mathfrak{i}_2\\
\mathfrak{h}'\psi_5-\mathfrak{h}_{12}\theta_5 &= \psi\mathfrak{i}_1'\\
\mathfrak{h}_{12}\psi_5'-\mathfrak{h}\theta_5' &= \psi\mathfrak{i}_2'\\
\mathfrak{h}_{12}\theta_5-\mathfrak{h}\psi_5 &= \psi\mathfrak{i}_1
\end{aligned}\right\};$$

$$\left.\begin{aligned}
\Delta'\theta_7^2-2\Delta_{12}\theta_7\psi_7+\Delta\psi_7^2 &= \mathfrak{g}^2\\
\Delta'\psi_7'^2-2\Delta_{12}\theta_7'\psi_7'+\Delta\theta_7'^2 &= \mathfrak{g}'^2\\
\Delta'\chi_7^2-2\Delta_{12}\chi_7\phi_7+\Delta\phi_7^2 &= \mathfrak{h}^2\\
\Delta'\phi_7'^2-2\Delta_{12}\phi_7'\chi_7'+\Delta\chi_7'^2 &= \mathfrak{h}'^2
\end{aligned}\right\};$$

ϕ_2	See $\theta_7\rho_7$	See $\lambda_7\chi_7$	$\rho_7\lambda_7+\psi_5\psi_5'\Delta$	See $\theta_7\chi_2$
	υ_2	See $\lambda_7\rho_7$	See $\lambda_7\phi_7'$	See $\theta_7\omega_2'$
		ω_2	See $\rho_7\psi_7'$	$\lambda_7\chi_2+$ $\mathfrak{g}_{12}\theta_5+\phi\theta_5\Delta_{12}$
			χ_2'	$\upsilon_2\lambda_7'-$ $\mathfrak{i}_2\phi+\phi\psi_5\Delta_{12}$
				ψ_7

v_2	ω_2	χ_2'	ψ_7	λ_7'	θ_7'	ϕ_7	
elation.	$\lambda_7\phi_2+\theta_5\phi\Delta$	$\lambda_7 v_2+\psi_5\phi\Delta$	No relation.	$\lambda_7\psi_7+\mathfrak{g}\Phi$	$\lambda_7\lambda_7'+\mathfrak{g}_{12}\Phi-\Phi^2\Delta_{12}$	$\phi_2\chi_2-\mathfrak{i}_1\theta_5-\theta_5^2\Delta_{12}$	ϕ $\mathfrak{i}_2\theta_5-$
e $\theta_7\chi_2'$	$\phi_2\psi_7'+\phi\psi_5'\Delta$	$v_2\psi_7'+\phi\theta_5'\Delta$	See $\theta_7\lambda_7'$	See $\theta_7\theta_7'$	$\lambda_7'\psi_7'+\mathfrak{g}\phi$	$\phi_2 v_2'-\mathfrak{i}_1\psi_5'-\theta_5\psi_5'\Delta_{12}$	ω $\mathfrak{i}_1'\theta_5-$
e $\lambda_7\chi_2'$	No relation.	No relation.	$\lambda_7\lambda_7'-\mathfrak{g}_{12}\phi-\phi^2\Delta_{12}$	See $\lambda_7\theta_7'$	No relation.	$v_2'\omega_2-\mathfrak{i}_2'\psi_5'-\psi_5'^2\Delta_{12}$	ω $\mathfrak{i}_1'\psi_5'-$
$-\psi\theta_5\Delta$	No relation.	$\rho_7\omega_2-\psi\psi_5'\Delta$	$\chi_2\phi_2+\mathfrak{i}_1\theta_5-\theta_5^2\Delta_{12}$	$v_2'\phi_2+\mathfrak{i}_2\theta_5-\theta_5\psi_5'\Delta_{12}$	$v_2'\omega_2+\mathfrak{i}_2'\psi_5'-\psi_5'^2\Delta_{12}$	No relation.	$\rho_7\phi$
e $\phi_7'\phi_2$	See $\chi_7\chi_2'$	See $\phi_7'\omega_2$	$\omega_2'\phi_2'+\mathfrak{i}_1\psi_5-\psi_5\theta_5'\Delta_{12}$	$v_2v_2'+\mathfrak{i}_1'\theta_5-\theta_5\theta_5'\Delta_{12}$	$\phi_2'\omega_2'+\mathfrak{i}_2'\theta_5'-\theta_5'\psi_5'\Delta_{12}$	See $\chi_7\rho_7'$	Se
elation.	$\rho_7\chi_2'+\psi\theta_5'\Delta$	No relation.	$\omega_2'v_2+\mathfrak{i}_2\psi_5-\psi_5^2\Delta_{12}$	$v_2\phi_2'+\mathfrak{i}_1\psi_5-\theta_5'\psi_5\Delta_{12}$	$\phi_2'\chi_2'+\mathfrak{i}_1'\theta_5'-\theta_5'^2\Delta_{12}$	$\rho_7\rho_7'+\mathfrak{k}_{12}\psi-\psi^2\Delta_{12}$	Se
e $\theta_7\rho_7$	See $\lambda_7\chi_7$	$\rho_7\lambda_7+\psi_5\psi_5'\Delta$	See $\theta_7\chi_2$	$\chi_2\lambda_7+\mathfrak{g}_{12}\theta_5$	$\lambda_7v_2'+\mathfrak{g}_{12}\psi_5'-\phi\psi_5'\Delta_{12}$	See $\chi_7\chi_2$	$\chi_2\rho$
v_2	See $\lambda_7\rho_7$	See $\lambda_7\phi_7'$	See $\theta_7\omega_2'$	See $\lambda_7\omega_2'$	$\phi_2'\lambda_7+\mathfrak{g}_{12}\theta_5'-\phi\theta_5'\Delta_{12}$	$\rho_7'\phi_2-\mathfrak{i}_1'\psi-\psi\theta_5\Delta_{12}$	Se
	ω_2	See $\rho_7\psi_7'$	$\lambda_7\chi_2+\mathfrak{g}_{12}\theta_5+\phi\theta_5\Delta_{12}$	See $\lambda_7 v_2'$	$\psi_7'v_2'+\mathfrak{g}\psi_5'$	$\chi_7v_2'-\mathfrak{k}\psi_5'$	ρ_7v_2'
		χ_2'	$v_2\lambda_7'-\mathfrak{i}_2\phi+\phi\psi_5\Delta_{12}$	See $\lambda_7\phi_2'$	$\phi_2'\psi_7'+\mathfrak{g}\theta_5'$	$\rho_7'\omega_2-\mathfrak{i}_2'\psi-\psi\psi_5\Delta_{12}$	Se
			ψ_7	No relation.	$\lambda_7'^2-\phi^2\Delta'$	$\chi_2^2-\theta_5^2\Delta'$	$\omega_2'\chi_2$
				λ_7'	No relation.	$v_2'\chi_2-\theta\,\psi_5'\Delta'$	$\omega_2'v_2'$
					θ_7'	$v_2'^2-\psi_5'^2\Delta'$	$\phi_2'v_2'$
						ϕ_7	No

e $\theta_7\chi_2$	$\chi_2\lambda_7+\mathfrak{g}_{12}\theta_5$	$\lambda_7\upsilon_2'+$ $\mathfrak{g}_{12}\psi_5'-\phi\psi_5'\Delta_{12}$	See $\chi_7\chi_2$	$\chi_2\rho_7-\mathfrak{h}_{12}\theta_5$	$\rho_7'\upsilon_2-$ $\mathfrak{i}_2'\psi+\psi\psi_5\Delta_{12}$	See $\theta_7\phi_7$	$\theta_7\rho_7+$ $\mathfrak{i}_2\theta_5+\theta_5\psi_5\Delta_{12}$	See $\lambda_7\phi_7$	$\rho_7'\lambda_7+$ $\mathfrak{i}_2\psi_5'+\psi_5\psi_5'\Delta_{12}$
e $\theta_7\omega_2'$	See $\lambda_7\omega_2'$	$\phi_2'\lambda_7+$ $\mathfrak{g}_{12}\theta_5'-\phi\theta_5'\Delta_{12}$	$\rho_7'\phi_2-$ $\mathfrak{i}_1'\psi-\psi\theta_5\Delta_{12}$	See $\phi_2\chi_7'$	See $\phi_7'\omega_2'$	$\phi_2\omega_2-\mathfrak{g}\psi$	See $\theta_7\chi_7'$	$\rho_7'\lambda_7+$ $\mathfrak{i}_1\theta_5+\theta_5\theta_5'\Delta_{12}$	See $\lambda_7\chi_7'$.
χ_2+ $+\phi\theta_5\Delta_{12}$	See $\lambda_7\upsilon_2'$	$\psi_7'\upsilon_2'+\mathfrak{g}\psi_5'$	$\chi_7\upsilon_2'-\mathfrak{h}\psi_5'$	$\rho_7\upsilon_2'-\mathfrak{h}_{12}\psi_5'$	$\rho_7'\chi_2-$ $\mathfrak{i}_1'\psi+\psi\theta_5\Delta_{12}$	$\chi_7\lambda_7'-$ $\mathfrak{i}_1\psi_5'+\theta_5\psi_5'\Delta_{12}$	See $\rho_7'\lambda_7$	See $\theta_7'\chi_7$	$\theta_7'\rho_7'-$ $\mathfrak{i}_2'\theta_5'+\theta_5'\psi_5'\Delta_{12}$
$\lambda_7'-$ $-\phi\psi_5\Delta_{12}$	See $\lambda_7\phi_2'$	$\phi_2'\psi_7'+\mathfrak{g}\theta_5'$	$\rho_7'\omega_2-$ $\mathfrak{i}_2'\psi-\psi\psi_5\Delta_{12}$	See $\rho_7\phi_2'$	$\phi_7'\phi_2'-\mathfrak{h}'\theta_5'$	$\rho_7'\lambda_7+$ $\mathfrak{i}_2'\psi_5+\psi_5\psi_5'\Delta_{12}$	$\chi_7'\lambda_7+$ $\mathfrak{i}_1'\psi_5+\theta_5'\psi_5\Delta_{12}$	See $\omega_2'\phi_2'$	See $\theta_7'\phi_7'$
ψ_7	No relation.	$\lambda_7'^2-\phi^2\Delta'$	$\chi_2^2-\theta_5^2\Delta'$	$\omega_2'\chi_2-\theta_5\psi_5\Delta'$	$\omega_2'^2-\psi_5^2\Delta'$	No relation.	No relation.	See $\lambda_7'\chi_2$	See $\lambda_7'\omega_2'$
	λ_7'	No relation.	$\upsilon_2'\chi_2-\theta\,\psi_5'\Delta'$	$\omega_2'\upsilon_2'-\theta_5\theta_5'\Delta'$	$\phi_2'\omega_2'-\theta_5'\psi_5\Delta'$	$\upsilon_2'\psi_7-\theta_5\phi\Delta'$	$\phi_2'\psi_7-\psi_5\phi\Delta'$	See $\theta_7'\chi_2$	See $\theta_7'\omega_2'$
		θ_7'	$\upsilon_2'^2-\psi_5'^2\Delta'$	$\phi_2'\upsilon_2'-\theta_5'\psi_5'\Delta'$	$\phi_2'^2-\theta_5'^2\Delta'$	$\lambda_7'\upsilon_2'-\phi\psi_5'\Delta'$	$\lambda_7'\phi_2'-\phi\theta_5'\Delta'$	No relation.	No relation.
			ϕ_7	No relation.	$\rho_7'^2-\psi^2\Delta'$	No relation.	$\rho_7'\chi_2-\psi\theta_5\Delta'$	No relation.	$\rho_7'\upsilon_2'-\psi\psi_5'\Delta'$
				ρ_7'	No relation.	See $\chi_7'\omega_2$	See $\chi_7'\chi_2$	See $\phi_7\phi_2'$	See $\chi_7'\upsilon_2'$
					χ_7'	$\rho_7'\omega_2'+\psi\psi_5\Delta'$	No relation.	$\rho_7'\phi_2'+\psi\theta_5'\Delta'$	No relation.
						χ_2	See $\rho_7'\psi_7$	See $\phi_7\lambda_7'$	$\rho_7'\lambda_7'+\psi_5\psi_5'\Delta'$
							ω_2'	See $\rho_7'\lambda_7'$	$\upsilon_2'\chi_2'+\mathfrak{g}'\psi$
								υ_2'	See $\rho_7'\theta_7'$

ψ_7	λ'_7	θ'_7	ϕ_7	ρ'_7	χ'_7	χ_2	ω'_2	υ'_2	ϕ'_2
relation.	$\lambda_7\psi_7+\mathfrak{g}\Phi$	$\lambda_7\lambda'_7+$ $\mathfrak{g}_{12}\Phi-\Phi^2\Delta_{12}$	$\phi_2\chi_2-$ $\mathfrak{i}_1\theta_5-\theta_5^2\Delta_{12}$	$\phi_2\omega_2-$ $\mathfrak{i}_2\theta_5-\theta_5\psi_5\Delta_{12}$	$\upsilon_2\omega'_2-$ $\mathfrak{i}_2\psi_5-\psi_5^2\Delta_{12}$	$\phi_2\psi_7-\mathfrak{g}\theta_5$	$\upsilon_2\psi_7-\mathfrak{g}\psi_5$	$\phi_2\lambda'_7-$ $\mathfrak{g}_2\theta_5+\theta_5\phi\Delta_{12}$	$\upsilon_2\lambda'_7-$ $\mathfrak{g}_{12}\psi_5+\psi_5\phi\Delta_{12}$
e $\theta_7\lambda'_7$	See $\theta_7\theta'_7$	$\lambda'_7\psi'_7+\mathfrak{g}\phi$	$\phi_2\upsilon'_2-$ $\mathfrak{i}_1\psi'_5-\theta_5\psi'_5\Delta_{12}$	$\omega_2\omega'_2-$ $\mathfrak{i}'_1\theta_5-\theta_5\theta'_5\Delta_{12}$	$\upsilon_2\phi'_2-$ $\mathfrak{i}_2\theta'_5-\theta'_5\psi_5\Delta_{12}$	See $\phi_2\lambda'_7$	$\upsilon_2\lambda'_7-\mathfrak{g}_{12}\psi_5$	$\omega_2\lambda'_7-\mathfrak{g}_{12}\psi'_5$	$\chi'_2\lambda'_7-\mathfrak{g}_{12}\theta'_5$
λ'_7- $-\phi^2\Delta_{12}$	See $\lambda_7\theta'_7$	No relation.	$\upsilon'_2\omega_2-$ $\mathfrak{i}'_2\psi'_5-\psi'^2_5\Delta_{12}$	$\omega_2\phi_2-$ $\mathfrak{i}'_1\psi'_5-\psi'_5\theta_5\Delta_{12}$	$\chi'_2\phi'_2-$ $\mathfrak{i}'_1\theta'_5-\theta'^2_5\Delta_{12}$	$\lambda_7\upsilon'_2-$ $\mathfrak{i}'_2\phi-\phi\psi'_5\Delta_{12}$	$\lambda'_7\chi'_2-$ $\mathfrak{g}_{12}\theta'_5-\phi\theta'_5\Delta_{12}$	See $\theta'_7\omega_2$	See $\theta'_7\chi'_2$
ϕ_2+ $-\theta_5^2\Delta_{12}$	$\upsilon'_2\phi_2+$ $\mathfrak{i}_2\theta_5-\theta_5\psi'_5\Delta_{12}$	$\upsilon'_2\omega_2+$ $\mathfrak{i}'_2\psi'_5-\psi'^2_5\Delta_{12}$	No relation.	$\rho_7\phi_7+\mathfrak{h}\psi$	$\rho_7\rho'_7-$ $\mathfrak{h}_{12}\psi-\psi^2\Delta_{12}$	$\phi_7\phi_2-\mathfrak{h}\theta_5$	$\rho_7\chi_2-$ $\mathfrak{i}_1\psi-\psi\theta_5\Delta_{12}$	See $\phi_7\omega_2$	$\rho_7\upsilon'_2-$ $\mathfrak{i}_2\psi-\psi\psi'_5\Delta_{12}$
ϕ'_2+ $-\psi_5\theta'_5\Delta_{12}$	$\upsilon_2\upsilon'_2+$ $\mathfrak{i}'_1\theta_5-\theta_5\theta'_5\Delta_{12}$	$\phi'_2\omega'_2+$ $\mathfrak{i}'_2\theta'_5-\theta'_5\psi'_5\Delta_{12}$	See $\chi_7\rho'_7$	See $\chi_7\chi'_7$	$\rho'_7\phi'_7+\mathfrak{h}'\psi$	See $\rho'_7\phi_2$	$\upsilon_2\rho'_7+\mathfrak{h}_{12}\psi_5$	See $\chi_7\phi'_2$	$\chi'_2\rho'_7+\mathfrak{h}_{12}\theta'_5$
υ_2+ $-\psi_5^2\Delta_{12}$	$\upsilon_2\phi'_2+$ $\mathfrak{i}_1\psi_5-\theta'_5\psi_5\Delta_{12}$	$\phi'_2\chi'_2+$ $\mathfrak{i}'_1\theta'_5-\theta'^2_5\Delta_{12}$	$\rho_7\rho'_7+$ $\mathfrak{h}_{12}\psi-\psi^2\Delta_{12}$	See $\rho_7\chi'_7$	No relation.	$\rho_7\omega'_2-$ $\mathfrak{i}_2\psi+\psi\psi_5\Delta_{12}$	$\upsilon_2\chi'_7+\mathfrak{h}'\psi_5$	$\rho_7\phi'_2-$ $\mathfrak{i}'_1\psi+\psi\theta'_5\Delta$	See $\chi'_7\chi'_2$
e $\theta_7\chi_2$	$\chi_2\lambda_7+\mathfrak{g}_{12}\theta_5$	$\lambda_7\upsilon'_2+$ $\mathfrak{g}_{12}\psi'_5-\phi\psi'_5\Delta_{12}$	See $\chi_7\chi_2$	$\chi_2\rho_7-\mathfrak{h}_{12}\theta_5$	$\rho'_7\upsilon_2-$ $\mathfrak{i}'_2\psi+\psi\psi_5\Delta_{12}$	See $\theta_7\phi_7$	$\theta_7\rho_7+$ $\mathfrak{i}_2\theta_5+\theta_5\psi_5\Delta_{12}$	See $\lambda_7\phi_7$	$\rho'_7\lambda_7+$ $\mathfrak{i}_2\psi'_5+\psi_5\psi'_5\Delta_{12}$
e $\theta_7\omega'_2$	See $\lambda_7\omega'_2$	$\phi'_2\lambda_7+$ $\mathfrak{g}_{12}\theta'_5-\phi\theta'_5\Delta_{12}$	$\rho'_7\phi_2-$ $\mathfrak{i}'_1\psi-\psi\theta_5\Delta_{12}$	See $\phi_2\chi'_7$	See $\phi'_7\omega'_2$	$\phi_2\omega_2-\mathfrak{g}\psi$	See $\theta_7\chi'_7$	$\rho'_7\lambda_7+$ $\mathfrak{i}_1\theta_5+\theta_5\theta'_5\Delta_{12}$	See $\lambda_7\chi'_7$.
χ_2+ $+\phi\theta_5\Delta_{12}$	See $\lambda_7\upsilon'_2$	$\psi'_7\upsilon'_2+\mathfrak{g}\psi'_5$	$\chi_7\upsilon'_2-\mathfrak{h}\psi'_5$	$\rho_7\upsilon'_2-\mathfrak{h}_{12}\psi'_5$	$\rho'_7\chi_2-$ $\mathfrak{i}'_1\psi+\psi\theta_5\Delta_{12}$	$\chi_7\lambda'_7-$ $\mathfrak{i}_1\psi'_5+\theta_5\psi'_5\Delta_{12}$	See $\rho'_7\lambda_7$	See $\theta'_7\chi_7$	$\theta'_7\rho'_7-$ $\mathfrak{i}'_2\theta'_5+\theta'_5\psi'_5\Delta_{12}$
λ'_7- $-\phi\psi_5\Delta_{12}$	See $\lambda_7\phi'_2$	$\phi'_2\psi'_7+\mathfrak{g}\theta'_5$	$\rho'_7\omega_2-$ $\mathfrak{i}'_2\psi-\psi\psi_5\Delta_{12}$	See $\rho_7\phi'_2$	$\phi'_7\phi'_2-\mathfrak{h}'\theta'_5$	$\rho'_7\lambda_7+$ $\mathfrak{i}'_2\psi_5+\psi_5\psi'_5\Delta_{12}$	$\chi'_7\lambda_7+$ $\mathfrak{i}'_1\psi_5+\theta'_5\psi_5\Delta_{12}$	See $\omega'_2\phi'_2$	See $\theta'_7\phi'_7$
ψ_7	No relation.	$\lambda'^2_7-\phi^2\Delta'$	$\chi_2^2-\theta_5^2\Delta'$	$\omega'_2\chi_2-\theta_5\psi_5\Delta'$	$\omega'^2_2-\psi_5^2\Delta'$	No relation.	No relation.	See $\lambda'_7\chi_2$	See $\lambda'_7\omega'_2$
	λ'_7	No relation.	$\upsilon'_2\chi_2-\theta\,\psi'_5\Delta'$	$\omega'_2\upsilon'_2-\theta_5\theta'_5\Delta'$	$\phi'_2\omega'_2-\theta'_5\psi_5\Delta'$	$\upsilon'_2\psi_7-\theta_5\phi\Delta'$	$\phi'_2\psi_7-\psi_5\phi\Delta'$	See $\theta'_7\chi_2$	See $\theta'_7\omega'_2$

$$\left.\begin{array}{l}
\Delta'\lambda_7^2 - 2\Delta_{12}\lambda_7\lambda_7' + \Delta\lambda_7'^2 = \mathfrak{g}_{12}^2 + (\Delta\Delta' - \Delta_{12}^2)\,\phi^2 \\
\Delta'\rho_7^2 - 2\Delta_{12}\rho_7\rho_7' + \Delta\rho_7'^2 = \mathfrak{h}_{12}^2 + (\Delta\Delta' - \Delta_{12}^2)\,\psi^2 \\
\Delta'\phi_2^2 - 2\Delta_{12}\phi_2\chi_2 + \Delta\chi_2^2 = \mathfrak{i}_1^2 + (\Delta\Delta' - \Delta_{12}^2)\,\theta_5^2 \\
\Delta'\upsilon_2^2 - 2\Delta_{12}\upsilon_2\omega_2' + \Delta\omega_2'^2 = \mathfrak{i}_2^2 + (\Delta\Delta' - \Delta_{12}^2)\,\psi_5^2 \\
\Delta'\omega_2^2 - 2\Delta_{12}\upsilon_2'\omega_2 + \Delta\upsilon_2'^2 = \mathfrak{i}_2'^2 + (\Delta\Delta' - \Delta_{12}^2)\,\psi_5'^2 \\
\Delta'\chi_2'^2 - 2\Delta_{12}\chi_2'\phi_2' + \Delta\phi_2'^2 = \mathfrak{i}_1'^2 + (\Delta\Delta' - \Delta_{12}^2)\,\theta_5'^2
\end{array}\right\};$$

The remainder are given in the following table, which is such that some equivalent value has been found for every product-pair of the twenty quantics in §66 which are of the third degree in the coefficients. There is no equivalent value for the product $\theta_7\lambda_7$; and the table is to be read, for instance, in the line λ_7,

there is no equivalent value for $\lambda_7\psi_7'$;

$$\lambda_7\chi_7 = \phi_2\omega_2 - \theta_5\psi_5'\Delta,$$

and for $\phi_2\omega_2$ a reference is made to the entry for $\lambda_7\chi_7$; and it will be noticed that the results are all of such a form that the difference of two of these product-pairs is expressible in terms of quantities each of which has at least one factor of the second degree.

71. Each of these relations between the solutions of the differential equations implies a syzygy between the corresponding concomitants of the system of two quantics, the concomitant u_x—whose leading coefficient is unity—being used, where necessary, to make the order and the class uniform in the syzygy.

72. Each of the solutions obtained determines a concomitant; the order and the class of each such function so determined are given in the following table:

ORDER IN x.	CLASS IN u.	LEADING COEFFICIENT.
1	1	$\theta_1,\ \theta_1',\ \theta_8,\ \theta_8',\ \mathfrak{S}_2,\ \mathfrak{S}_2',\ \mathfrak{S}_{12}.$
2	2	$\theta_5,\ \theta_5',\ \psi_5,\ \psi_5'.$
3	3	$\phi_2,\ \phi_2',\ \upsilon_2,\ \upsilon_2',\ \omega_2,\ \omega_2',\ \chi_2',\ \chi_2,\ \mathfrak{i}_1,\ \mathfrak{i}_1',\ \mathfrak{i}_2,\ \mathfrak{i}_2'.$
0	3	$\phi,\ \theta_7,\ \theta_7'.$
1	4	$\lambda_7,\ \lambda_7',\ \psi_7,\ \psi_7'.$
2	5	$\mathfrak{g},\ \mathfrak{g}_{12},\ \mathfrak{g}'.$
3	0	$\psi,\ \chi_7,\ \chi_7'.$
4	1	$\rho_7,\ \rho_7',\ \phi_7,\ \phi_7'.$
5	2	$\mathfrak{h},\ \mathfrak{h}_{12},\ \mathfrak{h}'.$

73. The symbolical values of the more important concomitants are as follow, the original quantics being

$$U = b_x u_\beta = c_x u_\gamma = \ldots\ldots, \quad U' = b'_x u_{\beta'} = c'_x u_{\gamma'} = \ldots\ldots,$$

and capital letters Θ_r, Φ_s, ... denoting the comcomitants which have the corresponding small letters θ_r, ϕ_s, ... with the same suffixes for leading coefficients:

$$\begin{cases} \Theta_1 = b_x u_\beta; \\ \Theta'_1 = b'_x u_{\beta'}; \end{cases}$$

$$\begin{cases} \Theta_8 = -\Theta_1 + u_x b_\beta; \\ \Theta'_8 = -\Theta'_1 + u_x b'_{\beta'}; \end{cases}$$

$$\begin{cases} \Theta_5 = -\Theta_1^2 + u_x c_\beta b_x u_\gamma; \\ \Psi_5 = -\Theta_1\Theta'_1 + u_x b_{\beta'} b'_x u_\beta; \\ \Psi'_5 = -\Theta_1\Theta'_1 + u_x b'_\beta b_x u_{\beta'}; \\ \Theta'_5 = -\Theta_1'^2 + u_x c'_{\beta'} b'_x u_{\gamma'}; \end{cases}$$

$$\begin{cases} \Theta_2 = -(bcu)(\beta\gamma x); \\ \Theta_{12} = -(bb'u)(\beta\beta' x); \\ \Theta'_2 = -(b'c'u)(\beta'\gamma' x); \end{cases}$$

$$\begin{cases} \Phi = -u_\beta u_{\beta'}(bb'u); \\ \Psi = -b_x b'_x(\beta\beta' x); \end{cases}$$

$$\begin{cases} \frac{1}{2}\Theta_7 = -d_\beta u_\gamma u_\delta(bcu); \\ \frac{1}{2}\Psi_7 = u_x c_{\beta'} u_\beta u_\gamma(bb'u) - c_x u_\beta u_\gamma u_{\beta'}(bb'u); \\ \frac{1}{2}\Psi'_7 = -u_x c'_\beta u_{\beta'} u_{\gamma'}(bb'u) + c'_x u_{\beta'} u_{\gamma'} u_\beta(bb'u); \\ \frac{1}{2}\Theta'_7 = -d'_{\beta'} u_{\gamma'} u_{\delta'}(b'c'u); \end{cases}$$

$$\begin{cases} \frac{1}{2}\Lambda_7 = \frac{1}{2}\Theta_8\Phi - d_{\beta'} u_{\delta'} u_\gamma(bcu)\,u_x; \\ \frac{1}{2}\Lambda'_7 = -\frac{1}{2}\Theta'_8\Phi - d'_\beta u_\delta u_{\gamma'}(b'c'u)\,u_x; \end{cases}$$

$$\begin{cases} \frac{1}{2}X_7 = b_\gamma c_x d_x(\beta\delta x); \\ \frac{1}{2}\Phi_7 = u_x b'_\gamma c_x d_x(\beta'\delta x) - b'_x c_x d_x u_\gamma(\beta'\delta x); \\ \frac{1}{2}\Phi'_7 = -u_x b_{\gamma'} c'_x d'_x(\beta\delta' x) + b_x c'_x d'_x u_{\gamma'}(\beta\delta' x); \\ \frac{1}{2}X'_7 = b'_{\gamma'} c'_x d'_x(\beta'\delta' x); \end{cases}$$

$$\left\{\begin{aligned} \tfrac{1}{2}\,\mathrm{P}_7 &= -\tfrac{1}{2}\,\Theta_8\Psi + b_{\beta'}b'_x c_x\,(\beta\gamma x)\,u_x;\\ \tfrac{1}{2}\,\mathrm{P}'_7 &= \quad\tfrac{1}{2}\,\Theta'_8\Psi + b'_\beta b_x c'_x\,(\beta'\gamma' x)\,u_x;\end{aligned}\right.$$

$$\left\{\begin{aligned} \tfrac{1}{2}\,\Phi_2 &= -\tfrac{1}{2}\,\Theta_8\Theta_5 + d_x u_\gamma\,(bcu)(\beta\delta x)\,u_x;\\ \tfrac{1}{2}\,\Phi'_2 &= -\tfrac{1}{2}\,\Theta'_8\Theta'_5 + d'_x u_{\gamma'}\,(b'c'u)(\beta'\delta' x)\,u_x;\end{aligned}\right.$$

$$\left\{\begin{aligned} \tfrac{1}{2}\,\Upsilon_2 &= -\tfrac{1}{2}\,\Theta_8\Psi_5 + b'_x u_\gamma\,(bcx)(\beta\beta' u)\,u_x;\\ \tfrac{1}{2}\,\Upsilon'_2 &= -\tfrac{1}{2}\,\Theta'_8\Psi'_5 - b_x u_{\gamma'}\,(b'c'x)(\beta\beta' u)\,u_x;\end{aligned}\right.$$

$$\left\{\begin{aligned} \tfrac{1}{2}\,\Omega_2 &= -\tfrac{1}{2}\,\Theta_8\Psi'_5 + c_x u_{\beta'}\,(bb'u)(\beta\gamma x)\,u_x;\\ \tfrac{1}{2}\,\Omega'_2 &= -\tfrac{1}{2}\,\Theta'_8\Psi_5 - c'_x u_\beta\,(bb'u)(\beta'\gamma' x)\,u_x;\end{aligned}\right.$$

$$\left\{\begin{aligned} \tfrac{1}{2}\,\mathrm{X}_2 &= -\tfrac{1}{2}\,\Theta'_8\Theta_5 - c_x u_\beta\,(bb'u)\,(\beta'\gamma x)\,u_x;\\ \tfrac{1}{2}\,\mathrm{X}'_2 &= -\tfrac{1}{2}\,\Theta_8\Theta'_5 + c'_x u_{\beta'}\,(bb'u)\,(\beta\gamma' x)\,u_x.^*\end{aligned}\right.$$

III.—*The Quadro-Linear Quantic.*

74. This may be taken in the form

$$\begin{aligned}&(ax_1^2 + bx_2^2 + cx_3^2 + 2fx_2x_3 + 2gx_3x_1 + 2hx_1x_2)\,u_1\\ +\,&(a'x_1^2 + b'x_2^2 + c'x_3^2 + 2f'x_2x_3 + 2g'x_3x_1 + 2h'x_1x_2)\,u_2\\ +\,&(a''x_1^2 + b''x_2^2 + c''x_3^2 + 2f''x_2x_3 + 2g''x_3x_1 + 2h''x_1x_2)\,u_3,\end{aligned}$$

which is symbolically represented by $a_x^2u_\alpha$. The characteristic equations are

$$\begin{aligned} D_1 = 2f\frac{\partial}{\partial b} + c\frac{\partial}{\partial f} + g\frac{\partial}{\partial h} + 2f'\frac{\partial}{\partial b'} + c'\frac{\partial}{\partial f'} + g'\frac{\partial}{\partial h'} - a'\frac{\partial}{\partial a''} - g'\frac{\partial}{\partial g''}&\\ - c'\frac{\partial}{\partial c''} + (2f'' - b')\frac{\partial}{\partial b''} + (c'' - f')\frac{\partial}{\partial f''} + (g'' - h')\frac{\partial}{\partial h''} &= 0,\\ \Delta = D_6 = 2f\frac{\partial}{\partial c} + b\frac{\partial}{\partial f} + h\frac{\partial}{\partial g} + 2f''\frac{\partial}{\partial c''} + b''\frac{\partial}{\partial f''} + h''\frac{\partial}{\partial g''} - a''\frac{\partial}{\partial a'}&\\ - b''\frac{\partial}{\partial b'} - h''\frac{\partial}{\partial h'} + (2f' - c'')\frac{\partial}{\partial c'} + (b' - f'')\frac{\partial}{\partial f'} + (h' - g'')\frac{\partial}{\partial g'} &= 0.\end{aligned}$$

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*I have not worked out in any detail the forms for *three lineo-linear* quantics; but it is interesting to see that the cubic determinant formed of the coefficients

$$\begin{matrix} a_1, h_1, g_1; & a'_1, h'_1, g'_1; & a''_1, h''_1, g''_1\\ a_2, h_2, g_2; & a'_2, h'_2, g'_2; & a''_2, h''_2, g''_2\\ a_3, h_3, g_3; & a'_3, h'_3, g'_3; & a''_3, h''_3, g''_3\end{matrix}$$

as the three "strata" (see Scott's "Determinants," Chap. VII), is a leading coefficient of a concomitant.

All the *simultaneous solutions can be expressed in terms of fifteen independent simultaneous solutions.*

75. It appears that, of the seventeen equations subsidiary to D_1, six integrals are at once given by a, g, c, a', g', c'; as in previous cases, we have a choice of variables of reference in g or a'.

The systems of integrals and the modified Δ-equations are formed as in the preceding cases, with the following results:

Quantities, being solutions of $D_1 = 0$ but not of $\Delta = 0$ and occurring in the equations, are

$$\left.\begin{array}{lll} \theta_2 = a', & \theta_6 = g', & \theta_8 = c' \\ \theta_{11} = f' + c'', & \theta_5 = g, & \theta_3 = c \end{array}\right\},$$

$$\left.\begin{array}{l} \theta_4 = (f,\ c \between a',\ a'') \\ \theta_7 = (\lambda,\ g' \between a',\ a'') \\ \theta_9 = (\mu,\ c' \between a',\ a'') \end{array}\right\}, \quad \left.\begin{array}{l} \phi_4 = (f,\ c \between g,\ -h) \\ \phi_7 = (\lambda,\ g' \between g,\ -h) \\ \phi_9 = (\mu,\ c' \between g,\ -h) \end{array}\right\},$$

$$\left.\begin{array}{l} \theta_{10} = (\rho,\ \mu,\ c' \between a',\ a'')^2 \\ \psi_{10} = (\rho,\ \mu,\ c' \between a',\ a'' \between g,\ -h) \\ \phi_{10} = (\rho,\ \mu,\ c' \between g,\ -h)^2 \end{array}\right\},$$

where the symbols λ, μ, ρ are defined by the equations

$$\lambda = \tfrac{1}{2}(h' - g''), \quad \mu = \tfrac{1}{3}(2f' - c''), \quad \rho = \tfrac{1}{3}(b' - 2f'').$$

Quantities, being solutions of both $D_1 = 0$ and $\Delta = 0$ and occurring in the equations, are, in addition to $y_1 = a$, given by

$$\left.\begin{array}{l} y_2 = (b,\ f,\ c \between a',\ a'')^2 \\ \xi_2 = (b,\ f,\ c \between a',\ a'' \between g,\ -h) \\ z_2 = (b,\ f,\ c \between g,\ -h)^2 \end{array}\right\}; \quad \left.\begin{array}{l} y_4 = (-h'',\ \lambda,\ g' \between a',\ a'')^2 \\ \xi_4 = (-h'',\ \lambda,\ g' \between a',\ a'' \between g,\ -h) \\ z_4 = (-h'',\ \lambda,\ g' \between g,\ -h)^2 \end{array}\right\};$$

$$\left.\begin{array}{l} y_5 = (-b'',\ \rho,\ \mu,\ c' \between a',\ a'')^3 \\ \eta_5 = (-b'',\ \rho,\ \mu,\ c' \between a',\ a'')^2(g,\ -h) \\ \xi_5 = (-b'',\ \rho,\ \mu,\ c' \between a',\ a'' \between g,\ -h)^2 \\ z_5 = (-b'',\ \rho,\ \mu,\ c' \between g,\ -h)^3 \end{array}\right\}; \quad \begin{array}{l} \left.\begin{array}{l} y_3 = ha' + ga'' \\ y_7 = h' + g'' \end{array}\right\}; \\ \left.\begin{array}{l} y_6 = (b' + f'',\ f' + c'' \between a',\ a'') \\ z_6 = (b' + f'',\ f' + c'' \between g,\ -h) \end{array}\right\}. \end{array}$$

And it is not difficult to see that, when θ_2 is the variable of reference, the

seventeen integrals of the equations subsidiary to D_1, being

$$y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ y_7;\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6,\ \theta_7,\ \theta_8,\ \theta_9,\ \theta_{10},\ \theta_{11},$$

are independent of one another.

76. The modified Δ-equations are

$\theta_2^2\Delta = \nabla$	$\theta_5^2\Delta = \nabla'$
$\nabla i = 2k$	$\nabla' i' = 2k'$
$\nabla k = y_2$	$\nabla' k' = z_2$
$\nabla l = y_3$	$\nabla' l' = -y_3$
$\nabla p = 2q$	$\nabla' p' = 2q'$
$\nabla q = y_4$	$\nabla' q' = z_4$
$\nabla r = 3s$	$\nabla' r' = 3s'$
$\nabla s = 2t$	$\nabla' s' = 2t'$
$\nabla t = y_5$	$\nabla' t' = z_5$
$\nabla v = y_6$	$\nabla' v' = z_6$
$\nabla \varkappa = \xi_2$	$\nabla' \varkappa' = \xi_2$
$\nabla \chi = \xi_4$	$\nabla' \chi' = \xi_4$
$\nabla \sigma = 2\omega$	$\nabla' \sigma' = 2\omega'$
$\nabla \omega = \eta_5$	$\nabla' \omega' = \xi_5$
$\nabla \tau = \xi_5$	$\nabla' \tau' = \eta_5$

where

$$\left.\begin{aligned}
\theta_5 &= l\theta_2, & \theta_2 &= l'\theta_5; \\
\theta_3 &= i\theta_2^2 & &= i'\theta_5^2 \\
\theta_4 &= k\theta_2 & &= \varkappa'\theta_5 \\
\phi_4 &= \varkappa\theta_2 & &= k'\theta_5 \\
\theta_6 &= p\theta_2^2 & &= p'\theta_5^2 \\
\theta_7 &= q\theta_2 & &= \chi'\theta_5 \\
\phi_7 &= \chi\theta_2 & &= q'\theta_5 \\
\theta_8 &= r\theta_2^3 & &= r'\theta_5^3 \\
\theta_9 &= s\theta_2^2 & &= \sigma'\theta_5^2 \\
\theta_{10} &= t\theta_2 & &= \tau'\theta_5 \\
\phi_9 &= \sigma\theta_2^2 & &= s'\theta_5^2 \\
\phi_{10} &= \tau\theta_2 & &= t'\theta_5 \\
\theta_{11} &= v\theta_2 & &= v'\theta_5 \\
\psi_{10} &= \omega\theta_2 & &= \omega'\theta_5
\end{aligned}\right\},$$

and the first nine equations are the independent equations for each of the variables of reference.

77. Taking first the set of nine independent equations in θ_2 as the variables of reference, eight independent integrals (the necessary number) are given by

$$\begin{aligned}
\delta_2 &= k^2 - iy_2 && = f^2 - bc, \\
\delta_4 &= q^2 - py_4 && = \lambda^2 + g'h'', \\
h_5 &= t^2 - sy_5 && = (\rho^2 + \mu b'',\ \rho\mu + b''c',\ \rho c' - \mu^2 \between a',\ a'')^2, \\
\phi_5 &= ry_5^2 - 3sty_5 + 2t^3 && = \boxed{\begin{array}{c|c|c|c} b''^2c' & -b''\rho c' & b''\mu c' & b''c'^2 \\ +3b''\rho\mu & +2b''\mu^2 & +2\rho^2c' & +3\rho\mu c' \\ +2\rho^3 & +\rho^2\mu & -\rho\mu^2 & -2\mu^3 \end{array}} \between a',\ a'')^3, \\
\varepsilon_2 &= ly_2 - ky_3 && = (gb - fh,\ gf - ch \between a',\ a''), \\
\varepsilon_4 &= ly_4 - qy_3 && = (-gh'' - \lambda h,\ g\lambda - hg' \between a',\ a''), \\
\eta_5 &= ly_5 - ty_3 && = (-b'',\ \rho,\ \mu,\ c' \between a',\ a'')^2(g,\ -h), \\
z_6 &= ly_6 - vy_3 && = (b' + f'',\ f' + c'' \between g,\ -h).
\end{aligned}$$

Here δ_2 and δ_4 are the discriminants of y_2 and y_4, regarded as binary quadratics in a' and a'' as variables; h_5 is the Hessian (with changed sign) of y_5 similarly regarded as a binary cubic, and ϕ_5 is its cubicovariant; ξ_2, ξ_4, η_5, z_6 are the Jacobians of y_3 with y_2, y_4, y_5, y_6 respectively, similarly regarded as binary quantics. And if we take

$$\delta_5 = b''^2c'^2 + 6b''\rho\mu c' - 4b''\mu^3 + 4\rho^3c' - 3\rho^2\mu^2,$$

being the discriminant of y_5, regarded as a binary cubic, we have

$$\phi_5^2 = \delta_5 y_5^2 + 4h_5^3,$$

so that δ_5 is a simultaneous solution and it may replace ϕ_5.

We thus have the result:

Every simultaneous solution of the two characteristic equations can be expressed in terms of the fifteen independent simultaneous solutions

$$y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ y_7,\ \delta_2,\ \delta_4,\ h_5,\ \delta_5,\ \xi_2,\ \xi_4,\ \eta_5,\ z_6;$$

and every concomitant of the quadro-linear quantic can be expressed in terms of the fifteen concomitants which have respectively these fifteen quantities for their leading coefficients.

78. Taking next the set of nine independent equations in θ_5 as the variable of reference, the corresponding eight integrals are

$$\begin{aligned}
\delta_2 &= k'^2 - i'z_2 = f^2 - bc, \\
\delta_4 &= q'^2 - p'z_4 = \lambda^2 + g'h'', \\
h_5' &= t'^2 - s'z_5 = (\rho^2 + \mu b'',\ \rho\mu + b''c',\ \rho c' - \mu^2 \between g,\ -h)^2, \\
[\phi_5' &= r'z_5^2 - 3s't'z_5 + 2t'^3, \text{ replaced as before by}] \\
\delta_5 &= \qquad\qquad = b''^2c'^2 + 6b''\rho\mu c' - 4b''\mu^3 + 4\rho^3c' - 3\rho^2\mu^2, \\
\xi_4 &= l'z_4 + q'y_3 = (-h''a' + \lambda a'',\ \lambda a' + ha'' \between g,\ -h), \\
\xi_2 &= l'z_2 + k'y_3 = (ba' + fa'',\ fa' + ca'' \between g,\ -h), \\
\xi_5 &= l'z_5 + t'y_3 = (-b'',\ \rho,\ \mu,\ c' \between g,\ -h)^2(a',\ a''), \\
y_6 &= l'z_6 + v'y_3 = (b' + f'',\ f' + c'' \between a',\ a'');
\end{aligned}$$

these quantities bearing similar relations to the quantities z, viewed as binary quantics in g and $-h$ as variables. Hence we see:

Every simultaneous solution of the two characteristic equations can also be expressed in terms of the fifteen independent simultaneous solutions

$$y_1,\ z_2,\ y_3,\ z_4,\ z_5,\ z_6,\ y_7,\ \delta_2,\ \delta_4,\ h_5',\ \delta_5,\ \xi_2,\ \xi_4,\ \xi_5,\ y_6;$$

and every concomitant of the quadro-linear quantic can also be expressed in terms of the fifteen concomitants which have respectively these fifteen quantities for their leading coefficients.

79 To obtain the order and the class of each of the concomitants, we may use either the method of developing operators in §7 and 8; or we may obtain the symbolical forms, the umbral coefficient-combinations being given in the accompanying table. For any one of them such as z_5, we first change the quantities b'', ρ, μ, c' (its coefficients regarded as a binary form) into umbral combinations, so that

	a_1^2	a_2^2	a_3^2	a_2a_3	a_3a_1	a_1a_2
α_1	a	b	c	f	g	h
α_2	a'	b'	c'	f'	g'	h'
α_3	a''	b''	c''	f''	g''	h''

$$\begin{aligned}
z_5 &= -g^3a_2^2\alpha_3 - g^2h(a_2^2\alpha_2 - 2a_2a_3\alpha_3) + gh^2(2a_2a_3\alpha_2 - a_3^2\alpha_3) - h^3a_3^2\alpha_2 \\
&= -(a_2g - a_3h)^2(\alpha_3g + \alpha_2h).
\end{aligned}$$

Now we have also

$$\begin{aligned}
a_2g - a_3h &= b_1\beta_1(a_2b_3 - a_3b_2) = b_1\beta_1(a_2b_3), \\
\alpha_3g + \alpha_2h &= d_1\delta_1(\alpha_3d_3 + \alpha_2d_2) = d_1\delta_1(d_\alpha - d_1\alpha_1),
\end{aligned}$$

so that

$$z_5 = -b_1\beta_1(a_2b_3)\,c_1\gamma_1(a_2c_3)\,d_1\delta_1\,(d_\alpha - d_1\alpha_1)$$

and therefore

$$Z_5 = -b_xc_xd_xu_\beta u_\gamma u_\delta\,(abu)(acu)\,d_\alpha u_x + b_xc_xu_\alpha u_\beta u_\gamma d_x^2u_\delta\,(abu)(acu).$$

The second term is seen to be resoluble, for $d_x^2 u_\delta$ is a factor; it is in fact equal to what is called $Y_1 Z_2$: and thus Z_5 effectively determines a concomitant $d_\alpha b_x c_x d_x u_\beta u_\gamma u_\delta (abu)(acu)$.

The symbolical expressions are here given for all except h_5, h_5' and δ_5, which are long and complicated in their symbolical form. The order in x and the class in u are 8 and 4, 4 and 6, 4 and 6 respectively for these three; and for the others are immediately evident from an inspection of their values:

$$
\begin{aligned}
Y_1 &= y_1 x_1^2 u_1 + \ldots = a_x^2 u_\alpha, \\
Y_3 &= y_3 x_1^4 u_1^2 + \ldots = -Y_1^2 + a_\beta a_x b_x^2 u_\alpha u_x, \\
Y_7 &= y_7 x_1^2 u_1 + \ldots = -Y_1 + a_x u_\alpha a_\alpha, \\
Y_2 &= y_2 x_1^6 u_1^3 + \ldots = -Y_1^3 - 2Y_1 Y_3 + a_\beta a_\gamma b_x^2 c_x^2 u_\alpha u_x^2, \\
Y_4 &= y_4 x_1^6 \quad + \ldots = a_\beta a_x b_x^2 c_x^2 (\alpha\gamma x), \\
Y_5 &= y_5 x_1^8 u_1 + \ldots = -2Y_1 Y_4 + a_\beta a_\gamma b_x^2 c_x^2 d_x^2 (\alpha\delta x)\, u_x, \\
Y_6 &= y_6 x_1^4 u_1^2 + \ldots = -Y_3 - Y_1 Y_7 - Y_1^2 + a_\alpha a_\beta b_x^2 u_x^2, \\
Z_2 &= z_2 x_1^2 u_1^5 + \ldots = b_x c_x u_\alpha u_\beta u_\gamma (abu)(acu), \\
Z_4 &= z_4 x_1^3 u_1^3 + \ldots = c_\beta a_x b_x c_x u_\alpha u_\gamma (abu), \\
Z_5 &= z_5 x_1^4 u_1^6 + \ldots = Y_1 Z_2 - d_\alpha b_x c_x d_x u_\beta u_\gamma u_\delta (abu)(acu)\, u_x, \\
Z_6 &= z_6 x_1 u_1^2 + \ldots = a_\alpha b_x u_\beta (abu), \\
\Delta_2 &= \delta_2 u_1^4 \quad + \ldots = -\tfrac{1}{2} u_\alpha u_\beta (abu)^2, \\
\Delta_4 &= \delta_4 x_1^4 u_1^2 + \ldots = \tfrac{1}{4} Y_7^2 - \tfrac{1}{2} a_x b_x (abu)(\alpha\beta x)\, u_x, \\
\Xi_2 &= \xi_2 x_1^3 u_1^3 + \ldots = a_\gamma b_x c_x^2 u_\alpha u_\beta (abu), \\
\Xi_4 &= \xi_4 x_1^6 u_1^3 + \ldots = -\tfrac{1}{2} Y_7 Y_3 + a_x b_x^2 c_x u_\gamma (acu)(\alpha\beta x)\, u_x, \\
\Xi_5 &= \xi_5 x_1^5 u_1^4 + \ldots = \tfrac{2}{3} Y_3 Z_6 + b_x c_x d_x^2 u_\beta u_\gamma (abu)(acu)(\alpha\delta x), \\
\mathrm{H}_5 &= \eta_5 x_1^8 u_1^4 + \ldots = \tfrac{2}{3} Y_3 Y_6 + 3Y_1 Y_2 + 3Y_1^2 Y_3 - Y_1^4 - a_\beta a_\gamma d_\alpha b_x^2 c_x^2 d_x u_\delta u_x^3.
\end{aligned}
$$

80. The operators which serve for the full development of the concomitants in powers of the variables from the leading coefficients are:

$$
\left.
\begin{aligned}
D_3 = 2h\frac{\partial}{\partial a} + b\frac{\partial}{\partial h} + f\frac{\partial}{\partial g} + 2h''\frac{\partial}{\partial a''} + b''\frac{\partial}{\partial h''} + f''\frac{\partial}{\partial g''} - b\frac{\partial}{\partial b'} \\
- f\frac{\partial}{\partial f'} - c\frac{\partial}{\partial c'} + (2h' - a)\frac{\partial}{\partial a'} + (b' - h)\frac{\partial}{\partial h'} + (f' - g)\frac{\partial}{\partial g'} \\
D_5 = 2g\frac{\partial}{\partial a} + f\frac{\partial}{\partial h} + c\frac{\partial}{\partial g} + 2g'\frac{\partial}{\partial a'} + f'\frac{\partial}{\partial h'} + c'\frac{\partial}{\partial g'} - b\frac{\partial}{\partial b''} \\
- f\frac{\partial}{\partial f''} - c\frac{\partial}{\partial c''} + (2g'' - a)\frac{\partial}{\partial a''} + (f'' - h)\frac{\partial}{\partial h''} + (c'' - g)\frac{\partial}{\partial g''}
\end{aligned}
\right\},
$$

so far as powers of the x-variables are concerned; and

$$\left.\begin{aligned} D_2 &= 2g'\frac{\partial}{\partial c'} + h'\frac{\partial}{\partial f'} + a'\frac{\partial}{\partial g'} + 2g''\frac{\partial}{\partial c''} + h''\frac{\partial}{\partial f''} + a''\frac{\partial}{\partial g''} - a''\frac{\partial}{\partial a} \\ &\quad - h''\frac{\partial}{\partial h} - b''\frac{\partial}{\partial b} + (2g - c'')\frac{\partial}{\partial c} + (h - f'')\frac{\partial}{\partial f} + (a - g'')\frac{\partial}{\partial g} \\ D_4 &= 2h'\frac{\partial}{\partial b'} + a'\frac{\partial}{\partial h'} + g'\frac{\partial}{\partial f'} + 2h''\frac{\partial}{\partial b''} + a''\frac{\partial}{\partial h''} + g''\frac{\partial}{\partial f''} - a'\frac{\partial}{\partial a} \\ &\quad - g'\frac{\partial}{\partial g} - c'\frac{\partial}{\partial c} + (2h - b')\frac{\partial}{\partial b} + (a - h')\frac{\partial}{\partial h} + (g - f')\frac{\partial}{\partial f} \end{aligned}\right\},$$

so far as powers of the u-variables are concerned.

82. Other solutions of the modified Δ-equations can be obtained, different in form but of course not algebraically independent; of these the most important are the set of three

$$\begin{aligned} \delta_{24} &= 2kq - iy_4 - py_2 = 2f\lambda + ch'' - bg', \\ \delta_{25} &= 2kt - sy_2 - iy_5 = (2f\rho - \mu b + b''c)\,a' + (2f\mu - bc' - \rho c)\,a'', \\ \delta_{45} &= 2qt - py_5 - sy_4 = (2\lambda\rho + b''g' + \mu h'')\,a' + (2\lambda\mu - g'\rho + c'h'')\,a'', \end{aligned}$$

which are respectively intermediaries between δ_2 and δ_4, δ_2 and δ_5, δ_4 and δ_5; and the set of functions of Jacobian form similar to those in §69.

83. The last statement is justified by the theorem:

The Jacobian of any two simultaneous solutions regarded as binary forms in a' and a'' is also a solution; and similarly for the Jacobians of solutions regarded as binary forms in g and $-h$.

The proof for the two cases is very much the same; taking it for the former, let U and V be two solutions of orders m and n respectively in a' and a'', and let J be their Jacobian. Then

$$\begin{aligned} mnJa' &= a'\left(\frac{\partial U}{\partial a''}\frac{\partial V}{\partial a'} - \frac{\partial U}{\partial a'}\frac{\partial V}{\partial a''}\right) \\ &= nV\frac{\partial U}{\partial a''} - mU\frac{\partial V}{\partial a''}. \end{aligned}$$

Now on looking at the groups of subsidiary quantities and comparing them with the equations, it appears that (1) for every simultaneous solution U, the associated quantity $\frac{\partial U}{\partial a''}$ satisfies $D_1 = 0$; hence, as a', U, V, $\frac{\partial U}{\partial a''}$, $\frac{\partial V}{\partial a''}$ all satisfy

$D_1 = 0$, it follows that J satisfies that equation; and (2) the modified Δ-equation is

$$\nabla\left(\frac{1}{ma'}\,\frac{\partial U}{\partial a''}\right) = U,$$

so that
$$\nabla J = V.U - U.V = 0,$$

and therefore J satisfies $\Delta = 0$. This proves the proposition for the former part; the latter part is similarly proved. Examples occur in ξ_2, ξ_4, ξ_5, η_5.

84. A set of dependent concomitants will thus be obtained having each as its leading coefficient a Jacobian in a' and a'' or in g and $-h$ of any two leading coefficients already obtained.

The relations among the various solutions will be similar to those previously obtained; only five examples will here be given, being those which connect the system of §77 with that of §78 immediately succeeding. They are

$$\begin{gathered} y_2 z_2 = \xi_2^2 - \delta_2 y_3^2, \\ y_4 z_4 = \xi_4^2 - \delta_4 y_3^2, \\ \left.\begin{aligned} y_5\xi_5 &= \eta_5^2 - h_5 y_3^2 \\ z_5\eta_5 &= \xi_5^2 - h_5' y_3^2 \end{aligned}\right\}, \\ h_5 h_5' = g_5^2 + \frac{1}{4}\,\delta_5 y_3^2, \end{gathered}$$

where g_5, an intermediate between h_5 and h_5', being equal to

$$\left(\rho^2 + \mu b'',\ \frac{1}{2}\,\rho\mu + \frac{1}{2}\,b''c',\ \rho c' - \mu^2 \between a',\ a'' \between g,\ -h\right),$$

is determined by the relation

$$2y_3^2 g_5 = \eta_5\xi_5 - y_5 z_5.$$

IV.—*The Cubo-Linear Quantic.*

85. This I take in the form

$$\begin{aligned} &u_1\,(ax_1^3 + 3hx_1^2x_2 + 3gx_1^2x_3 + 3bx_2^2x_1 + 3cx_3^2x_1 + 6fx_1x_2x_3 + ix_2^3 + 3jx_2^2x_3 + 3kx_2x_3^2 + lx_3^3) \\ &+ u_2\,(a'x_1^3 + 3h'x_1^2x_2 + 3g'x_1^2x_3 + 3b'x_2^2x_1 + 3c'x_3^2x_1 + 6f'x_1x_2x_3 + i'x_2^3 + 3j'x_2^2x_3 + 3k'x_2x_3^2 + l'x_3^3) \\ &+ u_3\,(a''x_1^3 + 3h''x_1^2x_2 + 3g''x_1^2x_3 + 3b''x_2^2x_1 + 3c''x_3^2x_1 + 6f''x_1x_2x_3 \\ &\qquad\qquad + i''x_2^3 + 3j''x_2^2x_3 + 3k''x_2x_3^2 + l''x_3^3) \end{aligned}$$

instead of a form such that the coefficient of u_1 is a uni-ternary cubic with coefficients as in Cayley's Third Memoir; the advantage being that all the analysis

of the quadro-linear quantic, as far as it goes, is valid here without any change. The characteristic equations are

$$
\begin{aligned}
D_1 = 3j\frac{\partial}{\partial i} + 2k\frac{\partial}{\partial j} + l\frac{\partial}{\partial k} + 3j'\frac{\partial}{\partial i'} + 2k'\frac{\partial}{\partial j'} + l'\frac{\partial}{\partial k'} + (3j'' - i)\frac{\partial}{\partial i''} \\
+ (2k'' - j')\frac{\partial}{\partial j''} + (l'' - k')\frac{\partial}{\partial k''} + 2f\frac{\partial}{\partial b} + c\frac{\partial}{\partial f} + g\frac{\partial}{\partial h} \\
+ 2f'\frac{\partial}{\partial b'} + c'\frac{\partial}{\partial f'} + g'\frac{\partial}{\partial h'} - a'\frac{\partial}{\partial a''} - g'\frac{\partial}{\partial g''} - c'\frac{\partial}{\partial c''} - l'\frac{\partial}{\partial l''} \\
+ (2f'' - b')\frac{\partial}{\partial b''} + (c'' - f')\frac{\partial}{\partial f''} + (g'' - h')\frac{\partial}{\partial h''} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta = D_6 = 3k\frac{\partial}{\partial l} + 2j\frac{\partial}{\partial k} + i\frac{\partial}{\partial j} + 3k''\frac{\partial}{\partial l''} + 2j''\frac{\partial}{\partial k''} + i''\frac{\partial}{\partial j''} + (3k' - l'')\frac{\partial}{\partial l'} \\
+ (2j' - k'')\frac{\partial}{\partial k'} + (i' - j'')\frac{\partial}{\partial j'} + 2f\frac{\partial}{\partial c} + b\frac{\partial}{\partial f} + h\frac{\partial}{\partial g} \\
+ 2f''\frac{\partial}{\partial c''} + b''\frac{\partial}{\partial f''} + h''\frac{\partial}{\partial g''} - a''\frac{\partial}{\partial a'} - b''\frac{\partial}{\partial b'} - h''\frac{\partial}{\partial h'} - i''\frac{\partial}{\partial i'} \\
+ (2f' - c'')\frac{\partial}{\partial c'} + (b' - f'')\frac{\partial}{\partial f'} + (h' - g'')\frac{\partial}{\partial g'} = 0.
\end{aligned}
$$

86. In addition to the quantities formed with a' as variable of reference which (§75) were for the quadro-linear quantic solutions of $D_1 = 0$ but not of $\Delta = 0$, viz. θ_r (where $r = 2, 3, \ldots, 10$) and the quantities which were for that same quantic solutions of $D_1 = 0$ and of $\Delta = 0$, viz. y_s (where $s = 1, 2, \ldots, 7$), all of which occupy similar positions in the construction of the equations for the present case—there are the additional θ-quantities, solutions of $D_1 = 0$ but not of $\Delta = 0$, given by

$$
\left\{\begin{aligned}
\theta_{12} &= l, \\
\theta_{13} &= ka' + la'', \\
\theta_{14} &= ja'^2 + 2ka'a'' + la''^2,
\end{aligned}\right.
\qquad
\left\{\begin{aligned}
\theta_{15} &= l', \\
\theta_{16} &= \varepsilon a' + l'a'', \\
\theta_{17} &= \pi a'^2 + 2\varepsilon a'a'' + l'a''^2, \\
\theta_{18} &= \sigma a'^3 + 3\pi a'^2 a'' + 3\varepsilon a'a''^2 + l'a''^3,
\end{aligned}\right.
$$

$$
\left\{\begin{aligned}
\theta_{19} &= k' + l'', \\
\theta_{20} &= (j' + k'')a' + (k' + l'')a'',
\end{aligned}\right.
$$

where $\varepsilon = \frac{1}{4}(3k' - l'')$, $\pi = \frac{1}{4}(2j' - 2k'')$, $\sigma = \frac{1}{4}(i' - 3j'')$; and there are the

additional y-quantities, solutions of $D_1 = 0$ and of $\Delta = 0$, given by

$$
\begin{aligned}
y_8 &= ia'^3 + 3ja'^2a'' + 3ka'a''^2 + la''^3 = (i, j, k, l \between a', a'')^3, \\
y_9 &= (-i'', \sigma, \pi, \varepsilon, l' \between a', a'')^4, \\
y_{10} &= (i' + j'', j' + k'', k' + l'' \between a', a'')^2.
\end{aligned}
$$

And the modified Δ-equations additional to those in the first column $(\theta_2^2\Delta = \nabla)$ of the table in §76 are

$$
\begin{aligned}
\nabla\,(\theta_{12}\theta_2^{-3}) &= 3\theta_{13}\theta_2^{-2}, \\
\nabla\,(\theta_{13}\theta_2^{-2}) &= 2\theta_{14}\theta_2^{-1}, \\
\nabla\,(\theta_{14}\theta_2^{-1}) &= y_8, \\
\nabla\,(\theta_{15}\theta_2^{-4}) &= 4\theta_{16}\theta_2^{-3}, \\
\nabla\,(\theta_{16}\theta_2^{-3}) &= 3\theta_{17}\theta_2^{-2}, \\
\nabla\,(\theta_{17}\theta_2^{-2}) &= 2\theta_{18}\theta_2^{-1}, \\
\nabla\,(\theta_{18}\theta_2^{-1}) &= y_9, \\
\nabla\,(\theta_{19}\theta_2^{-2}) &= 2\theta_{20}\theta_2^{-1}, \\
\nabla\,(\theta_{20}\theta_2^{-1}) &= y_{10}.
\end{aligned}
$$

87. Without proceeding to the formation of the modified Δ-equations when $g\,(=\theta_5)$ is taken as the variable of reference, or to the formation of the dependent equations in all the subsidiary quantities which arise in the two cases of θ_2 and of θ_5 respectively as variable of reference, these equations are sufficient to give the algebraically independent concomitants in terms of which all others can be expressed; but the simplest set of independent solutions, though complete functionally, form a system much less complete in point of form and syzygetic irreducibility than in preceding cases. This, however, is not important from our point of view, the purpose being the formation of an algebraically complete system. Such a system is given by:

(i) The quantics in a' and a'' as variables, viz.:

two of order zero in a' and a'', being y_1, y_7;
two of order one in a' and a'', being y_3, y_6;
three of order two in a' and a'', being y_2, y_4, y_{10};
two of order three in a' and a'', being y_5, y_8;
one of order four in a' and a'', being y_9;

(ii) the algebraically independent concomitants of each of these, taken singly, viz.:

the discriminant of each of the three y_2, y_4, y_{10};

the Hessian and the cubicovariant (or the discriminant, replacing the cubicovariant) of each of the two y_5, y_8;

the Hessian, the quadrinvariant and the cubicovariant (or the cubinvariant, replacing the cubicovariant) of y_9;

(iii) the Jacobian of y_3 with each of the seven y_6, y_2, y_4, y_{10}, y_5, y_8, y_9 in turn.

88. The total number in the system is thus 27, being less by three than the number of constants in the original quantic; this is the (§18) required number.

The order and the class of each of the concomitants determined by a leading coefficient will be determined subsequently (§§106–112) for the general biternary quantic.

V.—*The Cubo-Cubic Quantic.*

89. Without taking in separate detail the cases in order of simplicity after the last, viz. the lineo-quadratic, the lineo-cubic, quadro-quadratic and quadro-cubic, I pass on to the cubo-cubic, merely giving the equations on account of the mass of algebra. From the form in which the coefficients are taken for the present quantic, the equations apply to these omitted cases so far as in such omitted quantics the coefficients occur.

The quantic is taken in the form

$$\begin{aligned} & u_1^3 U_{00} \\ & + 3u_1^2 u_2 U_{10} + 3u_1^2 u_3 U_{01} \\ & + 3u_1 u_2^2 U_{20} + 6u_1 u_2 u_3 U_{11} + 3u_1 u_3^2 U_{02} \\ & + u_2^3 U_{30} + 3u_2^2 u_3 U_{21} + 3u_2 u_3^2 U_{12} + u_3^3 U_{03}; \end{aligned}$$

and the coefficients U in this quantic are

$$\begin{aligned} & ax_1^3 \\ & + 3hx_1^2 x_2 + 3gx_1^2 x_3 \\ & + 3bx_1 x_2^2 + 6fx_1 x_2 x_3 + cx_1 x_3^2 \\ & + ix_2^3 + 3jx_2^2 x_3 + 3kx_2 x_3^2 + lx_3^3, \end{aligned}$$

the literal coefficients a, h, g, ... being supposed to have the same indicative suffix as the quantity U in which they occur.

90. The subsidiary equations of the characteristic $D_1 = 0$ are

$$\frac{da_{00}}{0} = \frac{dh_{00}}{g_{00}} = \frac{dg_{00}}{0} = \frac{db_{00}}{2f_{00}} = \frac{df_{00}}{c_{00}} = \frac{dc_{00}}{0} = \frac{di_{00}}{3j_{00}} = \frac{dj_{00}}{2k_{00}} = \frac{dk_{00}}{l_{00}} = \frac{dl_{00}}{0}$$

$$= \frac{da_{10}}{0} = \frac{dh_{10}}{g_{10}} = \frac{dg_{10}}{0} = \frac{db_{10}}{2f_{10}} = \frac{df_{10}}{c_{10}} = \frac{dc_{10}}{0} = \frac{di_{10}}{3j_{10}} = \frac{dj_{10}}{2k_{10}} = \frac{dk_{10}}{l_{10}} = \frac{dl_{10}}{0}$$

$$= \frac{da_{01}}{-a_{10}} = \frac{dh_{01}}{g_{01}-h_{10}} = \frac{dg_{01}}{-g_{10}} = \frac{db_{01}}{2f_{01}-b_{10}} = \frac{df_{01}}{c_{01}-f_{10}} = \frac{dc_{01}}{-c_{10}} = \frac{di_{01}}{3j_{01}-i_{10}} = \frac{dj_{01}}{2k_{01}-j_{10}} = \frac{dk_{01}}{l_{01}-k_{10}} = \frac{dl_{01}}{-l_{10}}$$

$$= \frac{da_{20}}{0} = \frac{dh_{20}}{g_{20}} = \frac{dg_{20}}{0} = \frac{db_{20}}{2f_{20}} = \frac{df_{20}}{c_{20}} = \frac{dc_{20}}{0} = \frac{di_{20}}{3j_{20}} = \frac{dj_{20}}{2k_{20}} = \frac{dk_{20}}{l_{20}} = \frac{dl_{20}}{0}$$

$$= \frac{da_{11}}{-a_{20}} = \frac{dh_{11}}{g_{11}-h_{20}} = \frac{dg_{11}}{-g_{20}} = \frac{db_{11}}{2f_{11}-b_{20}} = \frac{df_{11}}{c_{11}-f_{20}} = \frac{dc_{11}}{-c_{20}} = \frac{di_{11}}{3j_{11}-i_{20}} = \frac{dj_{11}}{2k_{11}-j_{20}} = \frac{dk_{11}}{l_{11}-k_{20}} = \frac{dl_{11}}{-l_{20}}$$

$$= \frac{da_{02}}{-2a_{11}} = \frac{dh_{02}}{g_{02}-2b_{11}} = \frac{dg_{02}}{-2g_{11}} = \frac{db_{02}}{2f_{02}-2b_{11}} = \frac{df_{02}}{c_{02}-2f_{11}} = \frac{dc_{02}}{-2c_{11}} = \frac{di_{02}}{3j_{02}-2i_{11}} = \frac{dj_{02}}{2k_{02}-2j_{11}} = \frac{dk_{02}}{l_{02}-2k_{11}} = \frac{dl_{02}}{-2l_{11}}$$

$$= \frac{da_{30}}{0} = \frac{dh_{30}}{g_{30}} = \frac{dg_{30}}{0} = \frac{db_{30}}{2f_{30}} = \frac{df_{30}}{c_{30}} = \frac{dc_{30}}{0} = \frac{di_{30}}{3j_{30}} = \frac{dj_{30}}{2k_{30}} = \frac{dk_{30}}{l_{30}} = \frac{dl_{30}}{0}$$

$$= \frac{da_{21}}{-a_{30}} = \frac{dh_{21}}{g_{21}-h_{30}} = \frac{dg_{21}}{-g_{30}} = \frac{db_{21}}{2f_{21}-b_{30}} = \frac{df_{21}}{c_{21}-f_{30}} = \frac{dc_{21}}{-c_{30}} = \frac{di_{21}}{3j_{21}-i_{30}} = \frac{dj_{21}}{2k_{21}-j_{30}} = \frac{dk_{21}}{l_{21}-k_{30}} = \frac{dl_{21}}{-l_{30}}$$

$$= \frac{da_{12}}{-2a_{21}} = \frac{dh_{12}}{g_{12}-2h_{21}} = \frac{dg_{12}}{-2g_{21}} = \frac{db_{12}}{2f_{12}-2b_{21}} = \frac{df_{12}}{c_{12}-2f_{21}} = \frac{dc_{12}}{-2c_{21}} = \frac{di_{12}}{3j_{12}-2i_{21}} = \frac{dj_{12}}{2k_{12}-2j_{21}} = \frac{dk_{12}}{l_{12}-2k_{21}} = \frac{dl_{21}}{-2l_{21}}$$

$$= \frac{da_{03}}{-3a_{12}} = \frac{dh_{03}}{g_{03}-3h_{12}} = \frac{dg_{03}}{-3g_{12}} = \frac{db_{03}}{2f_{03}-3b_{12}} = \frac{df_{03}}{c_{03}-3f_{12}} = \frac{dc_{03}}{-3c_{12}} = \frac{di_{03}}{3j_{03}-3i_{12}} = \frac{dj_{03}}{2k_{03}-3j_{12}} = \frac{dk_{03}}{l_{03}-3k_{12}} = \frac{dl_{03}}{-3l_{12}}$$

being ninety-nine in all; and therefore ninety-nine independent integrals need to be obtained.

The characteristic equation $D_6 = \Delta = 0$ has the operator Δ given by

$$
\begin{array}{lllll}
 & & + h_{00}\dfrac{\partial}{\partial g_{00}} & & + b_{00}\dfrac{\partial}{\partial f_{00}} \\
- a_{01}\dfrac{\partial}{\partial a_{10}} & - h_{01}\dfrac{\partial}{\partial h_{10}} & + (h_{10} - g_{01})\dfrac{\partial}{\partial g_{10}} & - b_{01}\dfrac{\partial}{\partial b_{10}} & + (b_{10} - f_{01})\dfrac{\partial}{\partial f_{10}} \\
 & & + h_{01}\dfrac{\partial}{\partial g_{01}} & & + b_{01}\dfrac{\partial}{\partial f_{01}} \\
-2a_{11}\dfrac{\partial}{\partial a_{20}} & -2h_{11}\dfrac{\partial}{\partial h_{20}} & + (h_{20} - 2g_{11})\dfrac{\partial}{\partial g_{20}} & -2b_{11}\dfrac{\partial}{\partial b_{20}} & + (b_{20} - 2f_{11})\dfrac{\partial}{\partial f_{20}} \\
- a_{02}\dfrac{\partial}{\partial a_{11}} & - h_{02}\dfrac{\partial}{\partial h_{11}} & + (h_{11} - g_{02})\dfrac{\partial}{\partial g_{11}} & - b_{02}\dfrac{\partial}{\partial b_{11}} & + (b_{11} - f_{02})\dfrac{\partial}{\partial f_{11}} \\
 & & + h_{02}\dfrac{\partial}{\partial g_{02}} & & + b_{02}\dfrac{\partial}{\partial f_{02}} \\
-3a_{21}\dfrac{\partial}{\partial a_{30}} & -3h_{21}\dfrac{\partial}{\partial h_{30}} & + (h_{30} - 3g_{21})\dfrac{\partial}{\partial g_{30}} & -3b_{21}\dfrac{\partial}{\partial b_{30}} & + (b_{30} - 3f_{21})\dfrac{\partial}{\partial f_{30}} \\
-2a_{12}\dfrac{\partial}{\partial a_{21}} & -2h_{12}\dfrac{\partial}{\partial h_{21}} & + (h_{21} - 2g_{12})\dfrac{\partial}{\partial g_{21}} & -2b_{12}\dfrac{\partial}{\partial b_{21}} & + (b_{21} - 2f_{12})\dfrac{\partial}{\partial f_{21}} \\
- a_{03}\dfrac{\partial}{\partial a_{12}} & - h_{03}\dfrac{\partial}{\partial h_{12}} & + (h_{12} - g_{03})\dfrac{\partial}{\partial g_{12}} & - b_{03}\dfrac{\partial}{\partial b_{12}} & + (b_{12} - f_{03})\dfrac{\partial}{\partial f_{12}} \\
 & & + h_{03}\dfrac{\partial}{\partial g_{03}} & & + b_{03}\dfrac{\partial}{\partial f_{03}}
\end{array}
$$

$$
\begin{array}{lllll}
+ 2f_{00}\dfrac{\partial}{\partial c_{00}} & & + i_{00}\dfrac{\partial}{\partial j_{00}} & + 2j_{00}\dfrac{\partial}{\partial k_{00}} & + 3k_{00}\dfrac{\partial}{\partial l_{00}} \\
+ (2f_{10} - c_{01})\dfrac{\partial}{\partial c_{10}} & - i_{01}\dfrac{\partial}{\partial i_{10}} & + (i_{10} - j_{01})\dfrac{\partial}{\partial j_{10}} & + (2j_{10} - k_{01})\dfrac{\partial}{dk_{10}} & + (3k_{10} - l_{01})\dfrac{\partial}{\partial l_{10}} \\
+ 2f_{01}\dfrac{\partial}{\partial c_{01}} & & + i_{01}\dfrac{\partial}{\partial j_{01}} & + 2j_{01}\dfrac{\partial}{\partial k_{01}} & + 3k_{01}\dfrac{\partial}{\partial l_{01}} \\
+ (2f_{20} - 2c_{11})\dfrac{\partial}{\partial c_{20}} & -2i_{11}\dfrac{\partial}{\partial i_{20}} & + (i_{20} - 2j_{11})\dfrac{\partial}{\partial j_{20}} & + (2j_{20} - 2k_{11})\dfrac{\partial}{\partial k_{20}} & + (3k_{20} - 2l_{11})\dfrac{\partial}{\partial l_{20}} \\
+ (2f_{11} - c_{02})\dfrac{\partial}{\partial c_{11}} & - i_{02}\dfrac{\partial}{\partial i_{11}} & + (i_{11} - j_{02})\dfrac{\partial}{\partial j_{11}} & + (2j_{11} - k_{02})\dfrac{\partial}{\partial k_{11}} & + (3k_{11} - l_{02})\dfrac{\partial}{\partial l_{11}} \\
+ 2f_{02}\dfrac{\partial}{\partial c_{02}} & & + i_{02}\dfrac{\partial}{\partial j_{02}} & + 2j_{02}\dfrac{\partial}{\partial k_{02}} & + 3k_{02}\dfrac{\partial}{\partial l_{02}} \\
+ (2f_{30} - 3c_{21})\dfrac{\partial}{\partial c_{30}} & -3i_{21}\dfrac{\partial}{\partial i_{30}} & + (i_{30} - 3j_{21})\dfrac{\partial}{\partial j_{30}} & + (2j_{30} - 2k_{21})\dfrac{\partial}{\partial k_{30}} & + (3k_{30} - 3l_{21})\dfrac{\partial}{\partial l_{30}} \\
+ (2f_{21} - 2c_{12})\dfrac{\partial}{\partial c_{21}} & -2i_{12}\dfrac{\partial}{\partial i_{21}} & + (i_{21} - 2j_{12})\dfrac{\partial}{\partial j_{21}} & + (2j_{21} - 2k_{12})\dfrac{\partial}{\partial k_{21}} & + (3k_{21} - 2l_{12})\dfrac{\partial}{\partial l_{21}} \\
+ (2f_{12} - c_{03})\dfrac{\partial}{\partial c_{12}} & - i_{03}\dfrac{\partial}{\partial i_{12}} & + (i_{12} - j_{03})\dfrac{\partial}{\partial j_{12}} & + (2j_{12} - k_{03})\dfrac{\partial}{\partial k_{12}} & + (3k_{12} - l_{03})\dfrac{\partial}{\partial l_{12}} \\
+ 2f_{03}\dfrac{\partial}{\partial c_{03}} & & + i_{03}\dfrac{\partial}{\partial j_{03}} & + 2j_{03}\dfrac{\partial}{\partial k_{03}} & + 3k_{03}\dfrac{\partial}{\partial l_{03}}
\end{array}
$$

91. Proceeding in the usual manner and forming the modified Δ-equations in solutions of the subsidiary equations of $D_1=0$, we first take a_{10} for variable of reference. The notation for these solutions of the subsidiary equations will be maintained as in the last case, viz. a solution of $D_1=0$ but not of $\Delta=0$ will be denoted by θ, and one which is simultaneously a solution of $D_1=0$ and $D=0$ arising in the modification of the Δ-equations will be denoted by y.

The quantities y which thus arise are:

$$
\begin{aligned}
y_1 &= a_{00},\\
y_2 &= (g_{00},\ h_{00} \between a_{01},\ a_{10}),\\
y_3 &= (c_{00},\ f_{00},\ b_{00} \between a_{01},\ a_{10})^2,\\
y_{14} &= (l_{00},\ k_{00},\ j_{00},\ i_{00} \between a_{01},\ a_{10})^3,\\
y_4 &= (g_{10},\ h_{10}-g_{01},\ -h_{01} \between a_{01},\ a_{10})^2,\\
y_5 &= (c_{10},\ 2f_{10}-c_{01},\ b_{10}-2f_{01},\ -b_{01} \between a_{01},\ a_{10})^3,\\
y_{15} &= (l_{10},\ 3k_{10}-l_{01},\ 3j_{10}-3k_{01},\ i_{10}-3j_{01},\ -i_{01} \between a_{01},\ a_{10})^4,\\
y_6 &= (a_{20},\ -2a_{11},\ a_{02} \between a_{01},\ a_{10})^2,\\
y_7 &= (g_{20},\ -2g_{11}+h_{20},\ g_{02}-2h_{11},\ h_{02} \between a_{01},\ a_{10})^3,\\
y_8 &= (c_{20},\ -2c_{11}+2f_{20},\ c_{02}-4f_{11}+b_{20},\ 2f_{02}-2b_{11},\ b_{02} \between a_{01},\ a_{10})^4,\\
y_{16} &= (l_{20},\ -2l_{11}+3k_{20},\ l_{02}-6k_{11}+3j_{20},\ 3k_{02}-6j_{11}+i_{20},\ 3j_{02}-2i_{11},\ i_{02} \between a_{01},\ a_{10})^5,\\
y_{17} &= (a_{30},\ -3a_{21},\ 3a_{12},\ -a_{03} \between a_{01},\ a_{10})^3,\\
y_{18} &= (g_{30},\ -3g_{21}+h_{30},\ 3g_{12}-3h_{21},\ -g_{03}+3h_{12},\ -h_{03} \between a_{01},\ a_{10})^4,\\
y_{20} &= (c_{30},\ -3c_{21}+2f_{30},\ 3c_{12}-6f_{21}+b_{30},\ -c_{03}+6f_{12}-3b_{21},\\
&\qquad -2f_{03}+3b_{12},\ -b_{03} \between a_{01},\ a_{10})^5,\\
y_{21} &= (l_{30},\ -3l_{21}+3k_{30},\ 3l_{12}-9k_{21}+3j_{30},\ -l_{03}+9k_{12}-9j_{21}+i_{30},\\
&\qquad -3k_{03}+9j_{12}-3i_{21},\ -3j_{03}+3i_{12},\ -i_{03} \between a_{01},\ a_{10})^6,\\
y_9 &= h_{10}+g_{01},\\
y_{10} &= (f_{10}+c_{01},\ b_{10}+f_{01} \between a_{01},\ a_{10}),\\
y_{22} &= (k_{10}+l_{01},\ j_{10}+k_{01},\ i_{10}+j_{01} \between a_{01},\ a_{10})^2,\\
y_{12} &= (h_{20}+g_{11},\ -h_{11}-g_{02} \between a_{01},\ a_{10}),\\
y_{11} &= (f_{20}+c_{11},\ b_{20}-c_{02},\ -b_{11}-f_{02} \between a_{01},\ a_{10})^2,\\
y_{23} &= (k_{20}+l_{11},\ 2j_{20}+k_{11}-l_{02},\ -j_{11}+2k_{02}+i_{20},\ -j_{02}-i_{11} \between a_{01},\ a_{10})^3,\\
y_{26} &= (h_{30}+g_{21},\ -h_{21}-g_{12},\ h_{12}+g_{03} \between a_{01},\ a_{10})^2,\\
y_{25} &= (f_{30}+c_{21},\ b_{30}-f_{21}-2c_{12},\ -2b_{21}-f_{12}+c_{03},\ b_{12}+f_{03} \between a_{01},\ a_{10})^3,\\
y_{24} &= (k_{30}+l_{21},\ 2j_{30}-2l_{12},\ i_{30}-3j_{21}-3k_{12}+l_{03},\ -2i_{21}+2k_{03},\ i_{12}+j_{03} \between a_{01},\ a_{10})^4,\\
y_{13} &= b_{20}+2f_{11}+c_{02},\\
y_{27} &= (b_{30}+2f_{21}+c_{12},\ -b_{21}-2f_{12}-c_{03} \between a_{01},\ a_{10}),\\
y_{28} &= (j_{20}+2k_{11}+l_{02},\ i_{20}+2j_{11}+k_{02} \between a_{01},\ a_{10}),\\
y_{29} &= (j_{30}+2k_{21}+l_{12},\ i_{30}+j_{21}-k_{12}-l_{03},\ -i_{21}-2j_{12}-k_{03} \between a_{01},\ a_{10})^2,\\
y_{19} &= i_{30}+3j_{21}+3k_{12}+l_{03}.
\end{aligned}
$$

The quantities θ which arise in the formation of the modified Δ-equations can be expressed in the following manner: Let any of the quantities y, which

are evidently binary quantics in a_{01}, a_{10}, be of degree r in those two variables; and let the operator

$$\frac{1}{r}\frac{\partial}{\partial a_{01}}$$

be denoted by δ so that the highest power in the derivative has a numerical coefficient unity. Conformably with this definition we have

$$\delta^2 y = \delta . \delta y = \frac{1}{r-1}\frac{\partial}{\partial a_{01}}(\delta y) = \frac{1}{r(r-1)}\frac{\partial^2 y}{\partial a_{01}^2},$$

and so on. Then the quantities θ are:

$$\begin{aligned}
\theta_2 &= a_{10} \quad \text{(the variable of reference)},\\
\theta_3 &= \delta y_2 ,\\
\theta_5 , \theta_4 &= \delta y_3 , \delta^2 y_3 ,\\
\theta_7 , \theta_6 &= \delta y_4 , \delta^2 y_4 ,\\
\theta_{10}, \theta_9 , \theta_8 &= \delta y_5 , \delta^2 y_5 , \delta^3 y_5 ,\\
\theta_{12}, \theta_{11} &= \delta y_6 , \delta^2 y_6 ,\\
\theta_{15}, \theta_{14}, \theta_{13} &= \delta y_7 , \delta^2 y_7 , \delta^3 y_7 ,\\
\theta_{19}, \theta_{18}, \theta_{17}, \theta_{16} &= \delta y_8 , \delta^2 y_8 , \delta^3 y_8 , \delta^4 y_8 ,\\
\theta_{20} &= \delta y_{10},\\
\theta_{22}, \theta_{21} &= \delta y_{11}, \delta^2 y_{11},\\
\theta_{23} &= \delta y_{12},\\
\theta_{26}, \theta_{25}, \theta_{24} &= \delta y_{14}, \delta^2 y_{14}, \delta^3 y_{14},\\
\theta_{30}, \theta_{29}, \theta_{28}, \theta_{27} &= \delta y_{15}, \delta^2 y_{15}, \delta^3 y_{15}, \delta^4 y_{15},\\
\theta_{35}, \theta_{34}, \theta_{33}, \theta_{32}, \theta_{31} &= \delta y_{16}, \delta^2 y_{16}, \delta^3 y_{16}, \delta^4 y_{16}, \delta^5 y_{16},\\
\theta_{38}, \theta_{37}, \theta_{36} &= \delta y_{17}, \delta^2 y_{17}, \delta^3 y_{17},\\
\theta_{42}, \theta_{41}, \theta_{40}, \theta_{39} &= \delta y_{18}, \delta^2 y_{18}, \delta^3 y_{18}, \delta^4 y_{18},\\
\theta_{47}, \theta_{46}, \theta_{45}, \theta_{44}, \theta_{43} &= \delta y_{20}, \delta^2 y_{20}, \delta^3 y_{20}, \delta^4 y_{20}, \delta^5 y_{20},\\
\theta_{53}, \theta_{52}, \theta_{51}, \theta_{50}, \theta_{49}, \theta_{48} &= \delta y_{21}, \delta^2 y_{21}, \delta^3 y_{21}, \delta^4 y_{21}, \delta^5 y_{21}, \delta^6 y_{21},\\
\theta_{55}, \theta_{54} &= \delta y_{22}, \delta^2 y_{22},\\
\theta_{58}, \theta_{57}, \theta_{56} &= \delta y_{23}, \delta^2 y_{23}, \delta^3 y_{23},\\
\theta_{62}, \theta_{61}, \theta_{60}, \theta_{59} &= \delta y_{24}, \delta^2 y_{24}, \delta^3 y_{24}, \delta^4 y_{24},\\
\theta_{65}, \theta_{64}, \theta_{63} &= \delta y_{25}, \delta^2 y_{25}, \delta^3 y_{25},\\
\theta_{67}, \theta_{66} &= \delta y_{26}, \delta^2 y_{26},\\
\theta_{68} &= \delta y_{27},\\
\theta_{69} &= \delta y_{28},\\
\theta_{71}, \theta_{70} &= \delta y_{29}, \delta^2 y_{29},
\end{aligned}$$

wherein the first member on the left-hand side is equal to the first on the right-hand side, the second to the second, and so on.

92. These 70 quantities θ and 29 quantities y make up the necessary number of 99 integrals of the equations subsidiary to $D_1 = 0$. The 99 integrals, as taken above, are independent of one another.

93. In order to express in a concise form the modified Δ-equations, such a relation as

$$\theta_2\Delta\theta_{16} - 4\theta_{16}\Delta\theta_2 = 4\theta_{17}$$

will be represented by

$$[\theta_{16},\, 4] = 4\theta_{17};$$

with this notation the modified Δ-equations are:

$$[\theta_3\,,\, 1] = y_2 \;\ldots\ldots\; (\text{ii})$$

$$\left.\begin{array}{l}[\theta_4\,,\, 2] = 2\theta_5 \\ [\theta_5\,,\, 1] = y_3\end{array}\right\}\ldots\;(\text{iii})$$

$$\left.\begin{array}{l}[\theta_6\,,\, 2] = 2\theta_7 \\ [\theta_7\,,\, 1] = y_4\end{array}\right\}\ldots\;(\text{iv})$$

$$\left.\begin{array}{l}[\theta_8\,,\, 3] = 3\theta_9 \\ [\theta_9\,,\, 2] = 2\theta_{10} \\ [\theta_{10},\, 1] = y_5\end{array}\right\}\ldots\;(\text{v})$$

$$\left.\begin{array}{l}[\theta_{11},\, 2] = 2\theta_{12} \\ [\theta_{12},\, 1] = y_6\end{array}\right\}\ldots\;(\text{vi})$$

$$\left.\begin{array}{l}[\theta_{13},\, 3] = 3\theta_{14} \\ [\theta_{14},\, 2] = 2\theta_{15} \\ [\theta_{15},\, 1] = y_7\end{array}\right\}\ldots\;(\text{vii})$$

$$\left.\begin{array}{l}[\theta_{16},\, 4] = 4\theta_{17} \\ [\theta_{17},\, 3] = 3\theta_{18} \\ [\theta_{18},\, 2] = 2\theta_{19} \\ [\theta_{19},\, 1] = y_8\end{array}\right\}\ldots\;(\text{viii})$$

$$[\theta_{20},\, 1] = y_{10} \;\ldots\ldots\; (\text{x})$$

$$\left.\begin{array}{l}[\theta_{21},\, 2] = 2\theta_{22} \\ [\theta_{22},\, 1] = y_{11}\end{array}\right\}\ldots\;(\text{xi})$$

$$[\theta_{23},\, 1] = y_{12} \;\ldots\ldots\; (\text{xii})$$

$$\left.\begin{array}{l}[\theta_{24},\, 3] = 3\theta_{25} \\ [\theta_{25},\, 2] = 2\theta_{26} \\ [\theta_{26},\, 1] = y_{14}\end{array}\right\}\ldots\;(\text{xiv})$$

$$\left.\begin{array}{l}[\theta_{27},\, 4] = 4\theta_{28} \\ [\theta_{28},\, 3] = 3\theta_{29} \\ [\theta_{29},\, 2] = 2\theta_{30} \\ [\theta_{30},\, 1] = y_{15}\end{array}\right\}\ldots\;(\text{xv})$$

$$\left.\begin{array}{l}[\theta_{31},\, 5] = 5\theta_{32} \\ [\theta_{32},\, 4] = 4\theta_{33} \\ [\theta_{33},\, 3] = 3\theta_{34} \\ [\theta_{34},\, 2] = 2\theta_{35} \\ [\theta_{35},\, 1] = y_{16}\end{array}\right\}\ldots\;(\text{xvi})$$

$$\left.\begin{array}{l}[\theta_{36},\, 3] = 3\theta_{37} \\ [\theta_{37},\, 2] = 2\theta_{38} \\ [\theta_{38},\, 1] = y_{17}\end{array}\right\}\ldots\;(\text{xvii})$$

$$\left.\begin{array}{l}[\theta_{39},\, 4] = 4\theta_{40} \\ [\theta_{40},\, 3] = 3\theta_{41} \\ [\theta_{41},\, 2] = 2\theta_{42} \\ [\theta_{42},\, 1] = y_{18}\end{array}\right\}\ldots\;(\text{xviii})$$

$$\left.\begin{array}{l}[\theta_{43},\, 5] = 5\theta_{44} \\ [\theta_{44},\, 4] = 4\theta_{45} \\ [\theta_{45},\, 3] = 3\theta_{46} \\ [\theta_{46},\, 2] = 2\theta_{47} \\ [\theta_{47},\, 1] = y_{20}\end{array}\right\}\ldots\;(\text{xx})$$

$$\left.\begin{array}{l}[\theta_{48},\, 6] = 6\theta_{49} \\ [\theta_{49},\, 5] = 5\theta_{50} \\ [\theta_{50},\, 4] = 4\theta_{51} \\ [\theta_{51},\, 3] = 3\theta_{52} \\ [\theta_{52},\, 2] = 2\theta_{53} \\ [\theta_{53},\, 1] = y_{21}\end{array}\right\}\ldots\;(\text{xxi})$$

$$\left.\begin{array}{l}[\theta_{54},\, 2] = 2\theta_{55} \\ [\theta_{55},\, 1] = y_{22}\end{array}\right\}\ldots\;(\text{xxii})$$

$$\left.\begin{array}{l}[\theta_{56},\, 3] = 3\theta_{57} \\ [\theta_{57},\, 2] = 2\theta_{58} \\ [\theta_{58},\, 1] = y_{23}\end{array}\right\}\ldots\;(\text{xxiii})$$

$$\left.\begin{array}{l}[\theta_{59},\, 4] = 4\theta_{60} \\ [\theta_{60},\, 3] = 3\theta_{61} \\ [\theta_{61},\, 2] = 2\theta_{62} \\ [\theta_{62},\, 1] = y_{24}\end{array}\right\}\ldots\;(\text{xxiv})$$

$$\left.\begin{array}{l}[\theta_{63},\, 3] = 3\theta_{64} \\ [\theta_{64},\, 2] = 2\theta_{65} \\ [\theta_{65},\, 1] = y_{25}\end{array}\right\}\ldots\;(\text{xxv})$$

$$\left.\begin{array}{l}[\theta_{66},\, 2] = 2\theta_{67} \\ [\theta_{67},\, 1] = y_{26}\end{array}\right\}\ldots\;(\text{xxvi})$$

$$[\theta_{68},\, 1] = y_{27} \ldots\; (\text{xxvii})$$

$$[\theta_{69},\, 1] = y_{28} \ldots\; (\text{xxviii})$$

$$\left.\begin{array}{l}[\theta_{70},\, 2] = 2\theta_{71} \\ [\theta_{71},\, 1] = y_{29}\end{array}\right\}\ldots\;(\text{xxix})$$

Of these 69 equations 68 integrals are required, which, with the 29 simultaneous y-solutions already obtained, make up the (§18) 35 requisite number of $\left(\frac{1}{4}.4.5.4.5 - 3 =\right)$ 97 independent solutions.

94. Now regarding the indicated 25 groups of the complete system of equations, we find first that each group furnishes a certain number of solutions independent of one another and derivable only from that group; and the number of

solutions so furnished is less by unity than the number of equations contained in the group. In fact each group of itself determines the algebraically independent concomitants of the quantity y occurring in the group regarded as a binary quantic; the aggregate of these independent concomitants, excluding the quantity y, will be called the *binariant-system* of that binary quantic y. Thus the binariant-system of

$$u = (A_0, A_1, A_2, \ldots \between x_1, x_2)^n = a_x^n$$

is composed of

$$\frac{1}{2}(ab)^2 a_x^{n-2} b_x^{n-2},$$

$$\frac{1}{2}(ab)^2(ac)\, a_x^{n-3} b_x^{n-2} c_x^{n-1},$$

$$\frac{1}{2}(ab)^4 a_x^{n-4} b_x^{n-4},$$

$$\frac{1}{2}(ab)^4(ac)\, a_x^{n-5} b_x^{n-4} c_x^{n-1},$$

$$\frac{1}{2}(ab)^6 a_x^{n-6} b_x^{n-6},$$

and so on; in terms of these all the invariants and covariants of u can be algebraically expressed.

We thus have a number of solutions for each of the binary quantics y, and derivable from them in the case when y is of degree in a_{01} higher than unity; the number of *additional* solutions thus obtained is

1 from each of the set	$y_3, y_4, y_6, y_{11}, y_{22}, y_{26}, y_{29}$	$= 7$,	
2 " " " "	$y_5, y_7, y_{14}, y_{17}, y_{23}, y_{25}$	$= 12$,	
3 " " " "	$y_8, y_{15}, y_{18}, y_{24}$	$= 12$,	
4 " " " "	y_{16}, y_{20}	$= 8$,	
5 from	y_{21}	$= 5$,	$= 44$ in all.

As we have now used the equations in a group among themselves, we may now take only a single equation out of each group; it is most convenient for the purposes of integration to retain the last equation of the group.

95. We thus have 25 equations left, which will furnish 24 independent integrals.

The combination of any pair of equations leads to the Jacobian of the two quantities y occurring in those equations, regarded as binary quantics in a_{01}, a_{10}; thus from

$$[\theta_3, 1] = y_2, \quad [\theta_5, 1] = y_3$$

we derive a solution

$$\frac{1}{\theta_2}(\theta_3 y_3 - \theta_5 y_2)$$

easily seen to be the Jacobian of y_2 and y_3. Combining, then, equation (ii) in

turn with each of the equations last in the other groups, we have the necessary number of 24 independent solutions; and these are the 24 Jacobians of y_2 and each of the remaining quantities y which are not independent of a_{01} and a_{10}. Combining, then, all our solutions, we have

(i) the 29 quantities y,
(ii) the 44 derived through the binariant systems,
(iii) the 24 Jacobians,

making the total of 97, the required number.

96. The process of derivation from the 99 independent solutions of $D_1 = 0$ shows that the 97 simultaneous solutions are independent of one another; it follows from the theory that every simultaneous solution can be expressed in terms of them.

The order and class will be left undetermined until §§106–112, when they will be given for the general quantic.

97. If we take $\theta_3 = g_{00}$ as the variable of reference instead of $\theta_2 = a_{10}$ and proceed in the same way, we find a set of binary quantics which have $-h_{00}$ and $+g_{00}$ for their variables instead of a_{01} and a_{10}. The forms $y_1, y_2, y_9, y_{13}, y_{19}$ are unaltered; the remainder have their coefficients the same, and their modification consists in the mentioned change of variables.

The aggregate of independent simultaneous solutions is similarly constituted; we have in addition to the quantics their binariant systems, and the set of Jacobians taken of course with regard to the variables of the system of quantics.

We shall denote the quantics in these variables by z, so that if $y_\mu = (*\between a_0{}^1, a_{10})^\lambda$ for any index μ and degree λ, then z_μ will denote $(*\between -h_{00}, g_{00})^\lambda$ with the same coefficients as y_μ.

VI.—*The Ternary Quantic of Order n and Class m.*

98. The complete system of algebraically independent concomitants consists of three classes, the arrangement being made conveniently with regard to their leading coefficients.

The first class of leading coefficients consists of a number of binary quantics in a_{01} and a_{10} as variables; (the results will be enunciated only for this system, but it may be borne in mind that there is an equivalent system in g_{00} and $-h_{00}$ as variables).

The second class of leading coefficients is constituted by the several bina-

riant systems of each of the binary quantics in the first class taken singly; that is to say, the system of algebraically independent concomitants of any such quantic, the quantic itself excluded. It is evident that any quantic of the first degree only in a_{01} and a_{10}, or one of zero degree in them (that is to say, independent of them) will supply no leading coefficients to this class.

The third class of leading coefficients is constituted by the Jacobians of any one quantic which involves a_{01} and a_{10} with each of the others in turn which also involve these variables.

It thus appears that, if the first class be completely given, then the second and third classes can be derived from them.

99. Let y be any one of the leading coefficients of the first class, determining a concomitant of the form

$$y x_1^{\mu} u_1^{\nu} + \ldots .$$

A linear substitution is $x_1 = X_1$, $x_2 = X_3$, $x_3 = X_2$, which must leave the concomitant unchanged (save possibly as to sign), and must therefore leave y similarly unchanged. The effect of this substitution is to interchange coefficients of the quantic symmetrically associated with x_2 and x_3, u_2 and u_3; this interchange must therefore not affect y, a binary quantic in a_{01} and a_{10}. But a_{01} and a_{10} are interchanged by the substitution; hence the sole effect on y (except a possible change of sign) is to reverse the order of the terms.

Let $a_{01}^p A$ be the first term in y; the form of y is

$$a_{01}^p A + a_{01}^{p-1} a_{10} \Delta A + \frac{1}{2!} a_{01}^{p-2} a_{10}^2 \Delta^2 A + \ldots .$$

as follows from the differential equation $D_6 = \Delta = 0$ to be satisfied by y. The last term in the series will be $a_{10}^p A'$, where A' is the value of A when the above interchange is effected.

It thus appears that a knowledge of the single term $a_{01}^p A$ is sufficient to determine y. But now we proceed to show, what is indeed the ordinary inference in the theory of binary quantics, that a knowledge of A alone is sufficient to determine y.

For y is isobaric and therefore $a_{01}^p A$ and $a_{10}^p A'$ are of the same weight. Denoting the quantic by $a_x^n u_\alpha^m$ and using the assignation of weights in §4, we have

$$p = \frac{\text{weight of } A - \text{weight of } A'}{\text{weight of } a_{10} - \text{weight of } a_{01}} .$$

But the umbral values of a_{01} and a_{10} are $a_1^n \alpha_1^{m-1} \alpha_3$ and $a_1^n \alpha_1^{m-1} \alpha_2$ respectively, so that

$$\text{weight of } a_{10} - \text{weight of } a_{01} = \text{weight of } \alpha_2 - \text{weight of } \alpha_3$$
$$= 1,$$

and therefore $\qquad p = \text{weight of } A - \text{weight of } A'.$

Thus when A is known we can deduce p and so find y; and for this purpose it is really sufficient to take any term in A, obtain the corresponding term in A' by the interchange of coefficients of the quantic symmetrical with regard to u_2 and u_3, x_2 and x_3, and find the difference of the weights which determines p. For instance, in the case of the cubo-cubic we have in y_{24} as coefficient of the first term $k_{30} + l_{21}$; taking k_{30}, which is the coefficient of $x_2 x_3^2 u_2^3$ (disregarding numerical coefficients), we change it into the coefficient of $x_3 x_2^2 u_3^3$, i. e. into j_{03}, so that

$$\begin{aligned}
p &= \text{weight of } k_{30} - \text{weight of } j_{03} \\
&= \text{weight of } a_2 a_3^2 \alpha_2^3 - \text{weight of } a_2^2 a_3 \alpha_3^3 \\
&= \text{weight of } a_3 - \text{weight of } a_2 + 3\,(\text{weight of } \alpha_2 - \text{weight of } \alpha_3) \\
&= 1 + 3.1 = 4,
\end{aligned}$$

agreeing with the form there given.

Hence it appears that the theory of binary quantics applies, so that *if we know the coefficient of the highest power of a_{01} in y, and even nothing but this coefficient, we can obtain the value of y by pure differentiation with the operator Δ.*

The determination of the quantics y therefore resolves itself into a determination of the coefficients A of their first terms.

100. In the general biternary quantic we write $(s,\, t)_{\sigma,\,\tau}$ in place of $a_{r,\,s,\,t,\,\rho,\,\sigma,\,\tau}$ (with the conditions $n = r + s + t$, $m = \rho + \sigma + \tau$); so that $(s,\, t)_{\sigma,\,\tau}$ is the literal coefficient of $x_1^r x_2^s x_3^t u_1^\rho u_2^\sigma u_3^\tau$ and its umbral value is given by

$$(s,\, t)_{\sigma,\,\tau} = a_1^{n-s-t} a_2^s a_3^t \alpha_1^{m-\sigma-\tau} \alpha_2^\sigma \alpha_3^\tau .$$

The operator $\Delta (= D_6)$ of §59 is in this notation

$$\sum \left[\{ t\,(s+1,\, t-1)_{\sigma,\,\tau} - \sigma\,(s,\, t)_{\sigma-1,\,\tau+1} \} \frac{\partial}{\partial\,(s,\, t)_{\sigma,\,\tau}} \right],$$

the summation extending over all the values of s and t such that $s + t$ is not greater than n, and all values of σ and τ such that $\sigma + \tau$ is not greater than m.

101. *All the leading coefficients A of the binary quantics y, which are themselves leading coefficients of concomitants, are encluded in the formula*

$$\begin{aligned}
&(0,\, t)_{\tau-\lambda,\,\lambda} + \lambda\,(1,\, t-1)_{\tau-\lambda+1,\,\lambda-1} + \frac{\lambda\,(\lambda-1)}{2!}\,(2,\, t-2)_{\tau-\lambda+2,\,\lambda-2} + \dots \\
&\quad + \frac{\lambda\,(\lambda-1)}{2!}\,(\lambda-2,\, t-\lambda+2)_{\tau-2,\,2} + \lambda\,(\lambda-1,\, t-\lambda+1)_{\tau-1,\,1} + (\lambda,\, t-\lambda)_{\tau,\,0}
\end{aligned}$$

for the values
$$t=\lambda,\ \lambda+1,\ \ldots,\ n,$$
$$\tau=\lambda,\ \lambda+1,\ \ldots,\ m$$
for each single value of λ; *and the values of* λ *are*
$$\lambda=0,\ 1,\ 2,\ \ldots,\ m \text{ or } n$$
according as m *is less or is greater than* n. *Such a quantity we shall represent by* $A_{t,\tau,\lambda}$.

Having now the coefficient of the first term in the quantic y, it is necessary to detèrmine the degree p of that quantic in a_{01}. Taking any term of $A_{t,\tau,\lambda}$, say the first which is $(0,\ t)_{\tau-\lambda,\lambda}$, we make the substitution which interchanges the terms in x_2 and x_3, u_2 and u_3, before indicated; this interchange gives us $(t,\ 0)_{\lambda,\tau-\lambda}$, so that
$$\begin{aligned} p &= \text{weight of } (0,\ t)_{\tau-\lambda,\lambda} - \text{weight of } (t,\ 0)_{\lambda,\tau-\lambda} \\ &= \text{weight of } a_1^{n-t}a_3^t\alpha_1^{m-\tau}\alpha_2^{\tau-\lambda}\alpha_3^{\lambda} - \text{weight of } a_1^{n-t}a_2^t\alpha_1^{m-\tau}\alpha_2^{\lambda}\alpha_3^{\tau-\lambda} \\ &= t\,(\text{weight of } a_3 - \text{weight of } a_2) + (\tau-2\lambda)(\text{weight of } \alpha_2 - \text{weight of } \alpha_3) \\ &= t+\tau-2\lambda. \end{aligned}$$

Hence *the quantic, which may be denoted by* $y_{t,\tau,\lambda}$, is
$$y_{t,\tau,\lambda}=\left(\left\{1,\ \Delta,\ \frac{1}{2!}\Delta^2,\ \frac{1}{3!}\Delta^3,\ \ldots,\ \frac{1}{t+\tau-2\lambda!}\Delta^{t+\tau-2\lambda}\right\}A_{t,\tau,\lambda}\between a_{01},\ a_{10}\right)^{t+\tau-2\lambda},$$
with the foregoing limitations on the values of t, τ, λ; *the quantities* a_{01} *and* a_{10} *denoting* $(0,\ 0)_{0,1}$ *and* $(0,\ 0)_{1,0}$.

For instance, in the case of the cubo-cubic the several quantics

	t	τ	λ		t	τ	λ		t	τ	λ		t	τ	λ		t	τ	λ
y_1	0	0	0	y_5	2	1	0	y_{17}	0	3	0	y_{22}	3	1	1	y_{24}	3	3	1
y_2	1	0	0	y_{15}	3	1	0	y_{18}	1	3	0	y_{12}	1	2	1	y_{13}	2	2	2
y_3	2	0	0	y_6	0	2	0	y_{20}	2	3	0	y_{11}	2	2	1	y_{28}	3	2	2
y_{14}	3	0	0	y_7	1	2	0	y_{21}	3	3	0	y_{23}	3	2	1	y_{27}	2	3	2
				y_8	2	2	0	y_9	1	1	1	y_{26}	1	3	1	y_{29}	3	3	2
y_4	1	1	0	y_{16}	3	2	0	y_{10}	2	1	1	y_{25}	2	3	1	y_{19}	3	3	3

are given in the accompanying table. The reason that there is no entry here for $t,\ \tau,\ \lambda=0,\ 1,\ 0$ is that the corresponding coefficient of the first term is $(0,\ 0)_{01}$

which is one of the variables of the quantic. And this omission is general for the special group of values.

102. Let N be the total number of quantics in the above system for the $n^{\circ}m^{\text{ic}}$; then, adding unity on account of the single omission just referred to, we have

$$\begin{aligned} N+1 &= (m+1)(n+1) \text{ for the value } \lambda=0 \\ &\quad + mn \qquad\qquad\quad \text{"} \qquad \text{"} \quad \lambda=1 \\ &\quad + (m-1)(n-1) \quad \text{"} \qquad \text{"} \quad \lambda=2 \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots \\ &= (m+1)(n+1)+mn+(m-1)(n-1)+\ldots\ldots, \end{aligned}$$

the series containing either $n+1$ or $m+1$ terms, whichever is the smaller integer.

We thus have *all the leading coefficients of the first class*, and the number of them.

103. The *leading coefficients of the second class* are constituted by the members of the binariant systems of those of the first class considered as isolated binary quantics. The number of members in the binariant system of a binary quantic of degree p is $p-1$, provided p be not less than 2; but if p be zero or unity, there is no binariant system. Hence only those quantics of the previous class for which $t+\tau \geqq 2\lambda+2$ will furnish members of the second class of coefficients; and, if $y_{t,\tau,\lambda}$ be one such, the number it furnishes is $t+\tau-2\lambda-1$. Thus the total number of coefficients of this class is $N' = \Sigma\,(t+\tau-2\lambda-1)$ with the limitations

$$\begin{aligned} t+\tau &\geqq 2\lambda+2, \\ t &\geqq \lambda \leqq n, \\ \tau &\geqq \lambda \leqq m. \end{aligned}$$

104. The *leading coefficients of the third class* are constituted by the Jacobian of any one of the first class involving a_{01} and a_{10} (say $y_{1,0,0}$) with each of the others of the first class involving those quantities; and it is with regard to a_{01} and a_{10} that the Jacobians must be taken. Now, of the N quantics in the first class there are either $n+1$ or $m+1$ (whichever be the smaller integer) which do not involve a_{01} and a_{10}—they are in fact given by $t=\tau=\lambda$—and therefore the number of quantics, other than $y_{1,0,0}$, which do involve those quantities is

$$N'' = N-1-\begin{cases} n+1 \\ m+1 \end{cases},$$

taking in the last term the smaller of the two integers. Each such quantic

combined with $y_{1,0,0}$ furnishes a Jacobian, and therefore the number of leading coefficients of the third class is N''.

Hence the *total number of leading coefficients of all classes* is

$$N + N' + N'';$$

and each of these leading coefficients determines one of the system of algebraically independent concomitants of the biternary quantic, in terms of which any concomitant can be expressed.

105. As regards the equivalent system obtained by taking g_{00} as the variable of reference, exactly similar results are obtained. There is a set of N-quantics in $-h_{00}$ and g_{00} as variables and of the same degrees, so that we have

$$z_{t,\tau,\lambda} = \left(\left\{1, \Delta, \frac{1}{2!}\Delta^2, \ldots, \frac{1}{t+\tau-2\lambda!}\Delta^{t+\tau-2\lambda}\right\} A_{t,\tau,\lambda} \between -h_{00}, g_{00}\right)^{t+\tau-2\lambda}.$$

We have N' further coefficients of concomitants obtained by taking the various binariant systems of these z-quantics; and the third class of N'' Jacobians of any one of them, say $z_{1,0,0}$, with all the others, the variables being in the present case $-h_{00}, g_{00}$.

106. Having now obtained the leading coefficients, it is necessary to determine *the order and the class of each of the concomitants* so determined; for this purpose the symbolical method will be adopted.

We first change $A_{t,\tau,\lambda}$ into its symbolical form, which is easily found to be

$$a_1^{n-t}\alpha_1^{m-\tau}a_3^{t-\lambda}\alpha_2^{\tau-\lambda}(a_2\alpha_2 + a_3\alpha_3)^{\lambda}.$$

The effect of the operator Δ on $A_{t,\tau,\lambda}$ is to change $(s, t)_{\sigma,\tau}$, that is, $a_1^{n-s-t}a_2^s a_3^t \alpha_1^{m-\sigma-\tau}\alpha_2^{\sigma}\alpha_3^{\tau}$ into

$$t(s+1, t-1)_{\sigma,\tau} - \sigma(s, t)_{\sigma-1,\tau+1},$$

that is, into

$$ta_1^{n-s-t}a_2^{s+1}a_3^{t-1}\alpha_1^{m-\sigma-\tau}\alpha_2^{\sigma}\alpha_3^{\tau} - \sigma a_1^{n-s-t}a_2^s a_3^t\alpha_1^{m-\sigma-\tau}\alpha_2^{\sigma-1}\alpha_3^{\tau+1},$$

so that in the symbolical form the effect of the operator Δ is

$$\alpha_2\frac{\partial}{\partial a_3} - \alpha_3\frac{\partial}{\partial a_2},$$

and similarly for repetitions of the operator. Now when this symbolical Δ-form operates on $(a_2\alpha_2 + a_3\alpha_3)$ the result is zero, so that this quantity behaves like a constant for Δ; hence we have

$$y_{t,\tau,\lambda} = a_1^{n-t}\alpha_1^{m-\tau}(a_2\alpha_2 + a_3\alpha_3)^\lambda\left[\left(\left\{1,\ \Delta,\ \frac{\Delta^2}{2!},\ \ldots\right\} a_3^{t-\lambda}\alpha_2^{\tau-\lambda}\between a_{01},\ a_{10}\right)^{t+\tau-2\lambda}\right]$$
$$= a_1^{n-t}\alpha_1^{m-\tau}(a_2\alpha_2 + a_3\alpha_3)^\lambda(a_2a_{10} + a_3a_{01})^{t-\lambda}(\alpha_2a_{01} - \alpha_3a_{10})^{\tau-\lambda},$$

after substitution and reduction. And in this expression all the symbols except a_{01} and a_{10} are umbral.

107. We can at once derive from this form of $y_{t,\tau,\lambda}$ the order and the class to be associated with it, completing the elements of the concomitant. For every factor of the form $a_2\alpha_2 - a_3\alpha_3$—that is, $a_\alpha - a_1\alpha_1$—there are a single power of x and a single power of u occurring. For every factor of the form $a_2a_{10} + a_3a_{01}$—that is, $b_1^n\beta^{m-1}(a_2\beta_2 + a_3\beta_3) = b_1^n\beta_1^{m-1}(a_\beta - a_1\beta_1)$ there are a power $n+1$ of x and a power m of u occurring. For every factor of the form $\alpha_2a_{01} - \alpha_3a_{10}$—that is, $c_1^n\gamma_1^{m-1}(\alpha_2\gamma_3)$—there are a power $n+1$ of x and a power $m-1$ of u occurring. Hence the order in the x-variables is

$$n - t + \lambda + (t-\lambda)(n+1) + (n+1)(\tau-\lambda) = n(t+\tau-2\lambda) + n - \lambda + \tau;$$

and the class in the u-variables is

$$m - \tau + \lambda + (t-\lambda)\,m + (m-1)(\tau-\lambda) = m(t+\tau-2\lambda) + m - 2\tau + 2\lambda.$$

But by means of concomitants occurring earlier in the sequence, it is possible (as in §79) to take a linear combination of $y_{t,\tau,\lambda}$ and powers and products of those earlier concomitants such that the symbolical form of the concomitant determined by the linear combination is divisible by a power of u_x equal to $\lambda + (t-\lambda)$, i. e. by u_x^t; and thus $y_{t,\tau,\lambda}$ determines a concomitant which may be called congruent with

$$a_\alpha^\lambda a_x^{n-t}u_\alpha^{m-\tau}\overset{t-\lambda}{\Pi}\{a_\beta b_x^n u_\beta^{m-1}\}\overset{\tau-\lambda}{\Pi}\{c_x^n u_\gamma^{m-1}(\alpha\gamma x)\}.$$

Retaining, however, the simpler form of leading coefficient, the concomitant thence determined is

$$y_{t,\tau,\lambda}x_1^{n(t+\tau-2\lambda+1)+\tau-\lambda}u_1^{m(t+\tau-2\lambda+1)-2\tau+2\lambda} + \ldots\ldots;$$

and thus the order and the class of each concomitant of the first class of leading coefficients are determined.

108. Passing now to the second class of leading coefficients, constituted by the binariant systems of those in the first class, we know that they can be arranged in two sets which are respectively of the second and the third degrees

in the *coefficients* of the binary quantic coefficients, but which have not yet been given in any form either really or umbrally connected with the coefficients of biternary quantic. For this purpose let

$$y_{t,\tau,\lambda} = y_\xi^{p+q} = A a_\xi^p \theta_\xi^q,$$

where $p = t - \lambda$, $q = \tau - \lambda$, $A = a_1^{n-t} \alpha_1^{m-\tau} (a_2\alpha_2 + a_3\alpha_3)^\lambda$, $a_\xi = a_2 a_{10} + a_3 a_{01}$, $\theta_\xi = \alpha_2 a_{01} - \alpha_3 a_{10}$. Then the transvectants may be represented in the forms

$$(s)_{t,\tau,\lambda} = (yy')^{2s} y_\xi^{p+q-2s} y_\xi'^{p+q-2s}$$

for those of the second degree in the *coefficients* of y, and in the forms

$$(s')_{t,\tau,\lambda} = (yy')^{2s} (yy'')\, y_\xi^{p+q-2s-1} y_\xi'^{p+q-2s} y_\xi''^{p+q-1}$$

for those of the third degree in the *coefficients* of y.

109. Consider first the former class, those of the second degree in the coefficients of y. We have

$$y_\xi^{p+q} = A a_\xi^p \theta_\xi^q, \quad y_\xi'^{p+q} = B b_\xi^p \phi_\xi^q;$$

so that

$$y_\xi^{p+q-2s} (yy')^{2s} y_\xi'^{p+q-2s} = \Sigma A a_\xi^{p-\rho} \theta_\xi^{q-\sigma} (ay')^\rho (\theta y')^\sigma y_\xi'^{p+q-\rho-\sigma},$$

the numerator on the right-hand side extending to all values of ρ and σ such that $\rho + \sigma = 2s$. Also

$$y_\xi'^{p+q-\rho-\sigma} y_\eta'^\rho y_\zeta'^\sigma = \Sigma B b_\xi^{p-\nu-\mu} b_\eta^\nu b_\zeta^\mu \phi_\xi^{q-\rho-\sigma+\nu+\mu} \phi_\eta^{\rho-\nu} \phi_\zeta^{\sigma-\mu},$$

the summation on the right-hand side being for values $0, 1, \ldots, \rho$ of ν and values $0, 1, \ldots, \sigma$ of μ. Hence

$$\begin{aligned}(s)_{t,\tau,\lambda} &= (yy')^{2s} y_\xi^{p+q-2s} y_\xi'^{p+q-2s} \\ &= AB\Sigma a_\xi^{p-\rho} \theta_\xi^{q-\sigma} b_\xi^{p-\nu-\mu} (ab)^\nu (a\phi)^{\rho-\nu} (\theta b)^\mu (\theta\phi)^{\sigma-\mu} \phi_\xi^{q-\rho-\sigma+\nu+\mu},\end{aligned}$$

the summation extended to all values $0, 1, \ldots, \rho$ of ν; to all values $0, 1, \ldots, \sigma$ of μ, and to all values of ρ and σ such that $\rho + \sigma = 2s$.

When the various terms in this summation are completed into forms which contain the variables, so as to give the concomitant having $(s)_{t,\tau,\lambda}$ for its leading coefficient, it appears that they are of varying order in x and of varying class in u. But, as will be seen immediately, the difference between the order and the class is the same for all the terms; and therefore, on the multiplication of each

term by a power of u_x proper to the term, all the terms are made to be of the same order throughout and the same class throughout. And evidently this order and this class are the order and the class of the particular term, or aggregate of terms, in the summation, and they give when completed the highest order and the highest class of all the terms.

Considering, then, the term occurring under the sign of summation as the typical term, we have as in §107

the order in x-variables

$$
\begin{array}{ll}
= 2(n-t)+2\lambda & \text{from } AB \\
\quad +(p-\rho)(n+1) & \text{from } a_\xi^{p-\rho} \\
\quad +(q-\sigma)(n+1) & \text{from } \theta_\xi^{q-\sigma} \\
\quad +(p-\nu-\mu)(n+1) & \text{from } b_\xi^{p-\nu-\mu} \\
\quad +\rho-\nu & \text{from } (a\phi)^{\rho-\nu} \text{ for } (a\phi)=a_2\phi_2+a_3\phi_3=a_\phi-a_1\phi_1 \\
\quad +\mu & \text{from } (\theta b)^{\mu} \\
\quad +\sigma-\mu & \text{from } (\theta\phi)^{\sigma-\mu} \\
\quad +(q-\rho-\sigma+\nu+\mu)(n+1) & \text{from } \phi_\xi^{q-\rho-\sigma+\nu+\mu} \\
= 2(n-t+\lambda)+2(n+1)(p+q-\rho-\sigma)+\rho+\sigma-\nu; &
\end{array}
$$

while from the same typical term the order in u-variables

$$
\begin{array}{ll}
= 2(m-\tau)+2\lambda & \text{from } AB \\
\quad +(p-\rho)\,m & \text{from } a_\xi^{p-\rho} \\
\quad +(q-\sigma)(m-1) & \text{from } \theta_\xi^{q-\sigma} \\
\quad +(p-\nu-\mu)\,m & \text{from } b^{p-\nu-\mu} \\
\quad +(q-\rho-\sigma+\nu+\mu)(m-1) & \text{from } \phi_\xi^{q-\rho-\sigma+\nu+\mu} \\
\quad +\nu & \text{from } (ab)^{\nu} \\
\quad +(\rho-\nu) & \text{from } (a\phi)^{\rho-\nu} \\
\quad +\mu & \text{from } (\theta b)^{\mu} \\
= 2(m-\tau+\lambda)+2m(p+q-\rho-\sigma)-2q+2(\sigma+\rho)-\nu. &
\end{array}
$$

The difference of these two is at once seen to depend only upon n, m; t, τ, λ; and $\rho+\sigma$ ($=2s$) and is therefore the same for all terms.

The greatest value of each is given by the terms for which $\nu=0$, so that the order of the concomitant is

$$2(n-t+\lambda)+2(n+1)(p+q-2s)+2s,$$

and its class is

$$2(m-\tau+\lambda)+2m(p+q-2s)-2q+4s;$$

and the power of u_x, which must be associated with the foregoing typical term in its completed form, is u_x^{ν}. These are *the order and the class of the concomitant having as its leading coefficient* $(s)_{t,\tau,\lambda}$, the transvectant of the second degree and s^{th} rank of $y_{t,\tau,\lambda}$.

110. But, as in §§73 and 79, the preceding concomitant can, by the addition of suitable combinations of concomitants occurring earlier in the series, be reduced so as to leave only that single term which involves the highest power of u_x in the whole sum of terms which is the expression of the concomitant; and the concomitant can therefore be considered as congruent to the function given by that single term when the power of u_x has been removed from it.

Now the highest power of u_x occurring in the completed form of the typical term is

$$\begin{aligned}
&\lambda \text{ from } A, \text{ for } A \text{ gives when completed } a_x^{n-t}u_\alpha^{m-\tau}(a_\alpha u_x - a_x u_\alpha)^\lambda \\
&+\lambda \text{ from } B, \text{ similarly} \\
&+p-\rho \text{ from } a_\xi^{p-\rho} \\
&+p-\nu-\mu \text{ from } b_\xi^{p-\nu-\mu} \\
&+\rho-\nu \text{ from } (a\phi)^{\rho-\nu} \\
&+\mu \text{ from } (\theta b)^\mu \\
=\; & 2p+2\lambda-2\nu,
\end{aligned}$$

and the term or set of terms for which this is greatest are the terms given by $\nu=0$, so that the power of u_x to be removed is $u_x^{2p+2\lambda}$.

And the function to which the preceding concomitant is thus reduced is the sum of quantities

$$\left[a_\alpha^\lambda b_\beta^\lambda a_x^{n-t}u_\alpha^{m-\tau}b_x^{n-t}u_\beta^{m-\tau}a_\beta^\rho b_\alpha^\mu(\alpha\beta x)^{\sigma-\mu}\right]$$
$$\left[\prod^{p-\rho} c_x^n u_\gamma^{m-1}a_\gamma\right]\left[\prod^{q-\sigma} d_x^n u_\delta^{m-1}(\alpha\delta x)\right]\left[\prod^{p-\mu} e_x^n u_\epsilon^{m-1}b_\epsilon\right]\left[\prod^{q-\rho-\sigma+\mu} k_x^n u_\kappa^{m-1}(\beta\kappa x)\right]$$

for all values of ρ and σ such that $\rho+\sigma=2s$, the symbol Π implying the product of t quantities similar to those which immediately follow that symbol.

111. Similarly proceeding with the concomitant, whose leading coefficient $s')_{t,\tau,\lambda}$ is of the third degree in the coefficients of $y_{t,\tau,\lambda}$, we find that the order of the concomitant is

$$3(n-t+\lambda)+(n+1)(3p+3q-4s-2)+2s+1,$$

and that the class is

$$3(m-\tau+\lambda)+m(3p+3q-4s-2)-3q+4s+2$$

All the elements of the concomitants, determined by the second class of leading coefficients, have now been obtained.

112. Lastly, for the third class of coefficients constituted by the Jacobians, we take the Jacobians of $y_{1,0,0}$ with all the other quantics. The Jacobian of $y_{1,0,0}$ with $y_{t,\tau,\lambda}$ can be taken, with the preceding notation, in the form

$$\frac{A}{p+q}b_1^{n-1}\beta_1^m\{pa_\xi^{p-1}\theta_\xi^q(ab)+qa_\xi^p\theta_\xi^{q-1}(\theta b)\},$$

from which it appears that the order in x is

$$(p+b+1)(n+1)-2-t+\lambda,$$

and that the class in u is

$$(p+q+1)m+2-\tau-q+\lambda.$$

This completes the determination of the elements of all the concomitants in the algebraically independent system of the biternary $n^o m^{ic}$.

113. As a special case of the foregoing, serving to render the results obtained more precise, I add the elements of the system of the *quadro-quadratic*, represented by

$$\begin{array}{l} (a_{00}x_1^2+2h_{00}x_1x_2+2g_{00}x_1x_3+b_{00}x_2^2+2f_{00}x_2x_3+c_{00}x_3^2)\,u_1^2 \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +(a_{02}x_1^2+2h_{02}x_1x_2+2g_{02}x_1x_3+b_{02}x_2^2+2f_{02}x_2x_3+c_{02}x_3^2)\,u_3^2. \end{array}$$

The quantities y_μ are the same as those denoted by the same symbols in §91; the values of t, τ, λ are those to be associated with y_μ from the preceding general investigation; h_μ is the Hessian of y_μ, so that $s=1$ and ϕ_μ is its cubicovariant, for which also $s=1$; i_μ is the quadrinvariant of y_μ for which $s=2$; and $j_{2,\mu}$ is the Jacobian of y_2 and y_μ. The values of m and of p are the orders in x-variables and the classes in u-variables of the concomitants determined by the leading coefficients; and the necessary 33 concomitants (§§18 and 35) of the system for the quadro-quadratic have their elements as given in the following table:

t	τ	λ	FIRST CLASS OF LEADING COEFFICIENT.	m	p	SECOND CLASS OF LEADING COEFFICIENT.	m	p	THIRD CLASS OF LEADING COEFFICIENT.	m	p
0	0	0	y_1	2	2						
1	0	0	y_2	4	4						
2	0	0	y_3	6	6	h_3	2	8	$j_{2,3}$	5	8
1	1	0	y_4	7	4	h_4	4	4	$j_{2,4}$	6	6
2	1	0	y_5	9	6	h_5	8	8	$j_{2,5}$	8	8
						ϕ_5	12	12			
0	2	0	y_6	8	2	h_6	6	0	$j_{2,6}$	7	4
1	2	0	y_7	10	4	h_7	10	4	$j_{2,7}$	9	6
						ϕ_7	15	6			
2	2	0	y_8	12	6	h_8	14	8	$j_{2,8}$	11	8
						ϕ_8	21	12			
						i_8	4	4			
2	2	1	y_{11}	7	4	h_{11}	4	4	$j_{2,11}$	6	6
2	1	1	y_{10}	4	4				$j_{2,10}$	3	6
1	2	1	y_{12}	5	2				$j_{2,12}$	4	4
1	1	1	y_9	2	2						
2	2	2	y_{13}	2	2						

In terms of these concomitants every concomitant of the quadrato-quadratic can be expressed; the simplest cases of all appear to be

$$(Y_1 + Y_9 + Y_{13}) \div u_x^2 = \text{linear invariant } a_\alpha^2,$$
$$(Y_1 + Y_9) \div u_x \quad = \text{linear concomitant } a_\alpha a_x u_\alpha.$$

The following short abstract of the contents of the paper may prove useful for reference:

INTRODUCTION AND BIBLIOGRAPHY; SEE ALSO NOTE TO §60.

Part I. 1-3—The differential equations of ternariants.
4—Assignation of weights.
5-12—Expansion of concomitants in powers of variables, and determination of leading coefficients, of order m and of class p.
13—Equations satisfied by leading coefficients of different kinds of ternariants.
14—Determination of order and class from symbolized form of a leading coefficient, and determination of $m-p$ by inspection of its weight.
15-18—All the concomitants of a quantic can be algebraically expressed in terms of a finite number of independent concomitants.
16—Notation for the quantics, and values of the literal operators which occur in the differential equations.
17—Leading coefficients are simultaneous concomitants of a system of binary quantics.

Part II. 19-21—Algebraically complete system of concomitants of a *quadratic*.
22-32— " " " " " " *cubic*.
33—Symbolical representation of concomitants.
34—Modification of the complete system of the cubic.
35—Method of obtaining from the differential equations the number (§18) of concomitants necessary to form the complete system of the n^{tic}.
36-42—Algebraically complete system of a *quartic*.
43-45— " " " " a ternary n^{tic}.
46-52— " " " " *two quadratics*.
53-58— " " " " *three quadratics*.

Part III. 59—The literal operators for bipartite quantics.
60-64—System of a bipartite *lineo-linear* quantic.
65-73— " of *two lineo-linear* quantics.
74-84— " of *quadro-linear* quantic.
85-88— " of leading coefficients for *cubo-linear* quantic.
89-97— " " " " " *cubo-cubic* quantic.
98-105— " " " " " biternary n^{omic}.
106-112—Determination of the order and the class of the concomitants of the n^{omic} given by the leading coefficients.
113—Special case of the *quadro-quadratic*.

ERRATA.

P. 4, l. 4, for concomitants read *quantics*.
P. 12, l. 13, for also $\pm$ is read *also is* $\pm$.
P. 31, l. 11, for U_1 read U_0.
P. 32, l. 10, for $\frac{\Phi}{3}$ read Φ_3.